section "Introduction"
sec:intro

CONVEX DELTATOPES IN ALL DIMENSIONS, AND POLYHEDRAL SOAP FILMS

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Abstract. A deltatope is a (convex) polytope whose faces are all regular simplexes. We classify all deltatopes in all dimensions, as well as certain generalizations, like polytopes whose faces are deltatopes. One reason for interest in these deltatopes in that their duals give candidates for singularities in higher-dimensional soap films.

1. Introduction

A deltahedron is a polyhedron whose faces are all equilateral triangles. Although there are an infinite number of such polyhedra if we do not require convexity, only eight are strictly convex. Analogously, we define a deltatope to be a polytope in $\mathbb{R}^n$ whose faces are all regular $(n-1)$-simplices. We find that for $n = 4$ there are five convex deltatopes, but for $n > 4$ there are only three, namely the n-dimensional simplex and orthoplex and the bipyramid over the $(n-1)$-simplex. We then consider generalizations to polytopes whose faces of some lower dimension are simplices, and find for instance that the only polytopes in any dimension whose 7-faces are simplices are the three deltatopes and various orthoplicial pyramids.

We will use the Gauß map to construct geometric duals for our polytopes. For instance, the eight strictly convex deltahedra are dual to nets on the two-sphere consisting of geodesic arcs meeting in threes at equal angles. Such a net can be viewed as a minimal foam in the sphere, and the cone over it (to the origin in $\mathbb{R}^3$) is a stationary minimal surface. Seven of these cones, however, are unstable if we allow combinatorial changes at the vertex. Jean Taylor [Tay] constructed comparison surfaces with less area for each possibility other than the tetrahedron, thus showing that this is the only possible singularity for soap films in $\mathbb{R}^3$.

Key words and phrases. Deltahedron, soap-film, deltatope.
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section "Convex Polytopes and Deltatopes" sec:defns

If we now want to consider singularities of soap films in $\mathbb{R}^4$, we note that the tangent cone will be the cone over some minimal foam in S^3. If all the faces of such a foam are totally geodesic, then it is the geometric dual of some (convex) polytope in $\mathbb{R}^4$. Since the singularities in the foam are only the tetrahedral ones allowed by Taylor, this polytope has only tetrahedra for faces, and so is one of our five deltatopes. (Of course there might also be nonpolyhedral minimal cones, with curved faces, but we look only for the polyhedral possibilities.)

In every dimension, of course, the simplex is a deltatope. The paired calibration argument of Lawlor and Morgan [LM] shows that the cone over the dual simplex is always area-minimizing. Thus in every dimension, all deltatopes are among the candidates to give soap-film singularities.

In high enough dimensions, Ken Brakke [Bra1] has shown using calibrations that the other deltatopes, the orthoplex and bipyramid, also provide examples of minimizing cones: their duals are the hypercube and simplicial prism, and cones over these are minimizing. Thus to completely classify polyhedral soap-film singularities in higher dimensions, one must consider the generalization of deltatopes to polytopes in $\mathbb{R}^n$ whose $(n-1)$-faces are deltatopes, or equivalently, whose $(n-2)$-faces are regular simplices. We classify these as well, at least in the dimensions necessary for the soap-film problem. This suggests the family of further generalizations, to polytopes whose $(n-k)$-faces are regular simplices, for larger and larger k. The last interesting case is when $k = n-2$: we call these polytopes, whose 2-faces are all equilateral triangles, generalized deltatopes.

I would like to thank Rob Kusner and the other organizers of the Regional Geometry Institute at Five Colleges in 1991. The stimulating environment that summer (including discussions with Frank Morgan) led me to first develop and present there the classification of deltatopes in $\mathbb{R}^4$. I would also like to thank Ken Brakke, whose proof that the hypercube and simplicial-prism cones are minimizing in higher dimensions led me to consider the generalizations of deltatopes. Finally, I want to thank John Conway, who has taught me much about regular polytopes.

2. Convex Polytopes and Deltatopes

We will be examining polytopes with faces of prescribed shapes. Although some of our discussion would carry over to nonconvex polytopes, we are interested only in the convex case, and for us the word "polytope" will imply convexity. Thus we define a *polytope* D in $\mathbb{R}^n$ as a compact, positive-volume intersection of finitely many (at least $n+1$) closed half-spaces. Here and throughout, we let $m := n-1$; then the *faces* (or m-faces) of D are its intersections with the bounding hyperplanes of the defining half-spaces, where these have non-zero m-area. These faces are themselves polytopes in affine m-spaces, and their faces are called the $(m-1)$-faces of the original polytope D. We continue finding lower-dimensional faces in this manner; finally the 0-faces are the vertices or extreme points of D.

section “The Wythoff Construction and Deltatopes”
sec:wythoff

In many cases of interest, the vertices adjacent to a particular vertex v of D all lie in some hyperplane. (In particular, this will be true if all vertices of D lie on a sphere and all edges have equal length.) In this case, they are the vertices of a polytope in that hyperplane, called the *vertex figure* of D at v.

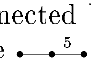
We define a *deltatope* in $\mathbb{R}^n$, $n > 2$ to be a (convex) polytope whose faces are all regular m-simplexes. (Here as always, we set $m := n - 1$.) More generally, a polytope D is *deltahedral in dimension* k if its k-faces are (congruent) regular simplices. We also call D then a *deltatope* $(m-k)$*-times removed*; for instance, a deltatope once-removed is a polytope whose faces are deltatopes.

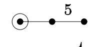
When $k = 2$, a polytope D deltahedral in dimension 2 is a polytope whose 2-faces are equilateral triangles. This we will call a *generalized deltatope*. Further generalizations are less interesting. We will say that any polytope with equal edge lengths is deltahedral in dimension 1. But this class includes an infinite number of families of nonrigid zonotopes, so it is too large to be interesting in our context. Any polytope is deltahedral in dimension 0.

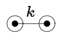
Note that if D is deltahedral in dimension k, then so are its j-faces (for $j \geq k$). And if D has a vertex figure V at some vertex, then V will be deltahedral in dimension $k - 1$. (So V, like D, is a deltatope $(m - k)$-times removed.) Of course, if P is deltahedral in dimension k, it is also deltahedral in any lower dimension.

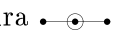
If all the k-faces of a polytope D are regular polytopes, then we will say D is *regular in dimension* k. We will see that almost all such polytopes are deltahedral in dimension $k - 1$. Of course, if D is deltahedral in dimension k, it is regular in dimension k.

3. The Wythoff Construction and Deltatopes

Most symmetric polytopes, including many (generalized) deltatopes, can be obtained from Coxeter reflection groups by the Wythoff construction (see [Cox]). A finite reflection group in $\mathbb{R}^n$ is generated by n mirror planes meeting at the origin. They cut out a simplex in the sphere S^m $(m = n - 1)$. Any two of the mirror planes meet, and must do so at an angle of $1/2k$ full turns, for some integer k. For most pairs of mirrors, $k = 2$, meaning that they are orthogonal. We represent the group by a Coxeter diagram, which has a node for each mirror generator. Nodes corresponding to mirrors meeting at an angle with $k > 2$ are connected by an edge labeled k (although we omit the label if $k = 3$). Thus for instance  represents the symmetry group of the icosahedron.

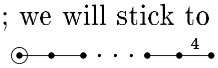
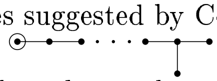
We now construct a polytope with this symmetry group by taking the convex hull of the images of any point p in space. The combinatorics of the resulting polytope depend only on which mirror planes contain the point p. We represent such a polytope by the Coxeter diagram with certain nodes—those corresponding to mirrors not containing p—circled. Thus for instance  represents the icosahedron itself. To get a truly n-dimensional polytope, we must circle at least one node in each component of the Coxeter diagram. Note that if there are two or more components in the diagram, the polytope is a cartesian product of polytopes in lower dimensions.

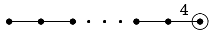
From the diagram for a polytope D, we can easily find the diagrams for its faces, by removing single nodes (and the incident edges) from the diagram. Of course, we only want to do this in the cases where the resulting diagram still has a circled node in each component. Repeating this process to find the lower-dimensional faces, we find that if the original diagram for D had more than one circled node, among its 2-faces is the $2k$-gon  (where we allow the possibility $k = 2$). Thus such a D cannot be a generalized deltatope.

We see that to construct a generalized deltatope D, we must start with a connected Coxeter diagram, and circle a single node. Furthermore, there cannot be an edge with a label $k > 3$ incident to this node. Also, if there is more than one edge incident to this node, D will not be deltahedral in dimension 3, since some of its 3-faces will be octahedra . We will examine below all such possibilities for D.

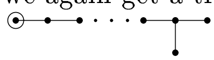
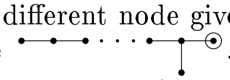
First, note that if D is given by a connected Coxeter diagram with a single end node circled, then D has vertex figure V, the same at every vertex. Removing the circled node from the diagram for D and circling instead the adjacent node gives us the diagram for V.

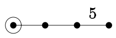
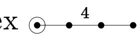
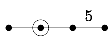
There are three regular polytopes which exist in every dimension. The regular *simplex* S_n in $\mathbb{R}^n$ (called α_n by Coxeter [Cox]) has diagram  and has $n+1$ vertices and $n+1$ faces. Each face is a simplex of lower dimension and is adjacent to every other face. The vertices can be taken to be cyclic permutations of $(1, 0, \dots, 0)$ in the affine subspace $\sum x_i = 1$ of $\mathbb{R}^{n+1}$. Then it is easy to see that cyclic permutations of $(n, -1 \dots, -1)$ are vectors normal to the faces; these are coordinates for the dual of S_n, combinatorially again a simplex. Taking dot products shows that the cosine of the exterior dihedral angle is $\hat{S}_n := -\frac{1}{n}$. Note that the interior dihedral angle increases with n, is exactly one-sixth turn for $n = 2$, is more than one-fifth turn for $n > 4$, and approaches one-quarter turn as $n \to \infty$.

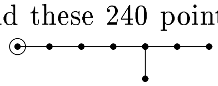
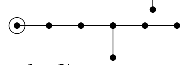
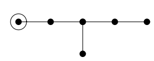
The regular *orthoplex* O_n (or β_n in [Cox]) in $\mathbb{R}^n$ is sometimes called the cross-polytope; we will stick to the names suggested by Conway [CS2]. It arises from the diagram  or from ; the latter expresses only half the symmetry. The orthoplex can be taken to have as vertices the cyclic permutations of $(\pm 1, 0, \dots, 0)$. There are 2^n simplicial faces, with normals $(\pm 1, \pm 1, \dots, \pm 1)$. The cosine of the dihedral angle between adjacent faces is thus $\hat{O}_n := 1 - \frac{2}{n}$. The interior dihedral angle again increases with n, is more than one-third turn for $n > 4$, and approaches one-half turn as $n \to \infty$.

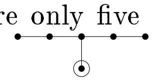
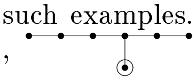
The third regular polytope is the *(hyper)cube* (γ_n in [Cox]), whose vertices can be taken to be $(\pm 1, \pm 1, \dots, \pm 1)$. Each of the $2n$ faces is itself a cube, and is orthogonal to the adjacent faces. The cube and orthoplex are dual to each other, and the diagram for the cube is that for the orthoplex with node at the other end circled: . The cube, of course, is not deltahedral.

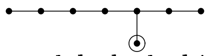
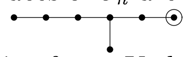
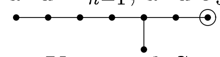
In low dimensions, of course, there are some further possible reflection groups; before we consider these, let us see what generalized deltatopes arise from the groups seen above.

The simplex S_n is of course deltahedral in dimension n. If we circle any other single node in its diagram , we get a truncation of the simplex. These are all deltahedral, but only in dimension two. The orthoplex O_n is a deltatope. If we circle a different node (but not one of the last two) in the diagram , we again get a truncation which is deltahedral only in dimension two. The diagram  represents the orthoplex with only half of its symmetry acknowledged. Moving the circle to a different node gives the same sequence of orthoplex truncations, except in the case . This is the *hemicube* H_n, whose vertices are alternate vertices of the cube (even sign changes of $(1, 1, \ldots, 1)$), and whose faces are simplices (at the missing corners of the cube) and hemicubes (at the faces of the cube). Since $H_3 = S_3$, the tetrahedron, all the hemicubes are deltahedral in dimension three (but not in dimension four).

Among the regular polyhedra in $\mathbb{R}^3$, the icosahedron I_3, like the tetrahedron and octahedron, is a deltahedron. The other regular polytopes in $\mathbb{R}^4$ are the tetraplex  (a deltatope I_4), its dual the dodecaplex, and the octaplex , which is deltahedral in dimension two and in fact equals tO_4. There is also one truncation  of the tetraplex which is deltahedral in dimension two.

The only three other finite irreducible reflection groups are the ones arising from the symmetries of the E_8 lattice (see [CS1]). This lattice can be taken to be the set of points in $\mathbb{R}^8$ whose coordinates are all half of either even integers or odd integers, with coordinate sum an even integer. The neighbors of the origin in this lattice are the permutations of $(\pm1, \pm1, 0, 0, 0, 0, 0, 0)$ and the even sign changes of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and these 240 points are the vertices of the Gossett polytope G_8 [Gos], with diagram . Then we can define G_{n-1} to be the vertex figure of G_n. So G_7 is  and G_6 is , while $G_5 = H_5$, G_4 is a truncated simplex, and G_3 is the triangular prism. Note that the faces of each Gosset polytope are simplexes and orthoplexes, so each is a deltatope once-removed. Coxeter uses the name k_{21} for what we are calling G_{k+4}; the numbers in his notation count the lengths of the branches of the diagram.

Moving the circle to a different node in the diagrams for these Gosset polytopes gives various related polytopes, all of which are deltahedral in dimension two. But only if the circle is placed on one of the other end nodes will the polytope be deltahedral in higher dimension. Thus there are only five such examples. We will use the names J_6, J_7 and J_8 for the polytopes ,  and section "Other Constructions for Deltatopes"

 respectively, which are 1_{22}, 1_{32} and 1_{42} in Coxeter's notation. They are deltahedral in dimension three: the faces of J_n are J_{n-1} and H_{n-1}, and $J_5 = H_5$. Similarly, K_7 and K_8 are the polytopes  and , or 2_{31} and 2_{41}. They are deltahedral in dimension four: K_n has faces K_{n-1} and S_{n-1}, while $K_6 = G_6$ and $K_5 = O_5$.

4. Other Constructions for Deltatopes

Given a polytope D in $\mathbb{R}^n$, we can define its (simple) truncation $t^j D$, $j < n$, as the convex polytope whose vertices are the centers of the j-faces of D. (Conway calls this the j^{th}-order *ambo-D*.) If D is deltahedral in dimension j, there is of course no question where the centers of these faces are geometrically. Such truncations are not in general deltahedral, though we have mentioned several instances of truncations which are deltahedral in dimension two. In particular, when D arises from the Wythoff construction by circling an end node in a Coxeter diagram, and there is a unique node j steps away in the graph, then $t^j D$ arises from circling this node in the same graph. We noted before that $t^j S_n$ $(j < n)$ and $t^j O_n$ $(j < n-2)$ are deltahedral in dimension two; for $n = 6, 7, 8$, we also have $t^j G_n$ $(j < n-4)$, tJ_n and tK_n, again deltahedral in dimension two.

If all the vertices of a polytope D lie on a sphere S^m, and if all its edges have equal length, greater than the radius of the sphere, we can construct a pyramid pD in $\mathbb{R}^{n+1}$ again with equal edge lengths. The faces of pD are D itself and pyramids on the faces of D. Because $pS_k = S_{k+1}$, if D is deltahedral in dimension k, so is pD. Similarly, we can construct a bipyramid bD, also deltahedral in dimension k, by gluing two pyramids together along their D faces. We should mention that if D is a deltatope, then so is bD, although from the previous sentence we would expect it only to be a deltatope once-removed.

Note that if the edges of D have length one, and the vertices lie on a sphere of radius r, then the vertices of pD all lie on a sphere of radius $1/(2\sqrt{1-r^2})$. The vertices of bD do not lie on a common sphere unless $r = 1/\sqrt{2}$.

As we noted above, the pyramid pS_m on a simplex is again a simplex S_n (where, as usual, $n = m + 1$). But the simplicial bipyramid is a polytope we have not seen before. We will write $B_n := bS_m$. This is a deltatope, for any n, and has $2n$ simplicial faces.

If we start with an orthoplex O_m, then we find the bipyramid $bO_m = O_n$ is just the next orthoplex. But the orthoplicial pyramid is a new polytope $P_n := pO_m$. In fact, we can continue taking repeated pyramids, and set $P_n^k := p(P_m^{k-1})$ with, of course, $P_n^0 := O_n$. In every case, $bP_m^k = P_n^k$. These pyramids can also be constructed by removing vertices from an orthoplex. There are $2n$ vertices of O_n. If we remove one vertex, and take the convex hull of the remaining vertices, we get P_n. We will refer to this operation as shaving off the one vertex, and we can write $P_n = sO_n$. If

% tab:examples

we also removed the opposite vertex, we would drop a dimension and get O_m, but if we remove any adjacent vertex, we get P_n^2. In general P_n^k is the convex hull of the remaining vertices if we remove k adjacent vertices of O_n. If we place O_n in the usual position with respect to cartesian coordinates, then P_n^k is obtained as half of P_n^{k-1}, sliced along one of the coordinate planes. (It is not clear if P_n^n should mean S_m, or its cone to the origin; we will only look at P_n^k for $k < n$.)

We can construct also pyramids and bipyramids on the polytopes G_m and H_m, although this only works for $m < 8$. For $m = 8$ both G_8 and H_8 have edge length equal to circumradius, so the pyramids would be flat.

We can also try shaving a vertex from other polytopes as we did from the orthoplex. If D has cospherical vertices and equal edge lengths, then it has a vertex figure V at any vertex v. If we remove v and let sD be the convex hull of the remaining vertices of D, then V will be one of the faces of sD. Faces of D not including v will still be faces of sD. A simplex including v is simply deleted. If F is nonsimplicial face involving v, then F is replaced by sF as a face of sD. For example, the faces of the Gosset polytope G_n are S_m and O_m, and all vertices are equivalent, with vertex figure G_m. If we remove one vertex to get sG_n, the vertex v is replaced by one face G_m, and the orthoplexes it was part of get replaced by pyramids $P_m = sO_m$. The resulting polytope is a deltatope twice-removed. In fact, we can simultaneously remove any number of vertices of G_n, as long as no two are adjacent, and still get a deltatope twice removed, with faces O_m, S_m, P_m and G_m. We will use sG_n to refer to any such polytope, while ssG_n will refer to the result of shaving a set of vertices including two (but not three) adjacent ones. These latter polytopes would be deltatopes thrice removed.

As another example of shaving, consider the icosahedral family. If we shave vertices off the icosahedron I_3, we get four polytopes sI_3 which are regular in dimension two: these result from shaving one vertex, two opposite vertices (giving the Archimedean pentagonal bipyramid), two nearly opposite vertices, or three nearly opposite vertices. The many shavings sI_4 of the tetraplex are polytopes with tetrahedral and icosahedral faces; among these is the snub octaplex (see [Cox]) first discovered by Gosset, and obtained by dividing the edges of the octaplex in the golden ratio. The second-order shavings ssI_4 include polytopes with faces sI_3; one notable one is Conway's grand antiprism. (This, along with the snub octaplex, is the only Archimedean polytope in $\mathbb{R}^4$ not obtainable from the Wythoff construction [Con].)

We list in Table 1 the various generalized deltatopes that we have constructed so far. The simplex, orthoplex, simplicial bipyramid, various orthoplicial pyramids, and hemicube exist in all dimensions. We conjecture that there are no generalized deltatopes in dimensions greater than eight, except those in the table. The various polytopes associated to the E_8 lattice exist only through dimension eight, and it seems that they probably cannot be used to build anything else in higher dimensions.

% sec:hedra
% tab:examples
section "Polytopes with regular faces"

The icosahedral family of polytopes only exists through dimension four. In particular low dimensions, there are various conincidences. For instance $G_6 = K_6$ (deltatopes once removed in $\mathbb{R}^6$ are deltahedral in dimension four), and $K_5 = O_5$, $J_5 = G_5 = H_5$. In $\mathbb{R}^4$, $K_4 = J_4 = S_4$, $H_4 = O_4$, $G_4 = tS_4$; also tO_4 is the octaplex.

Type	all n	$n \leq 8$ only	$n \leq 4$ only
Simplex	S		
Deltatope	B, O		I, bI
D. once-removed	P	G	pI, tI, sI
D. twice-removed	P^2	pG, bG, sG	psI, bsI, ssI
D. thrice-removed	P^3	ppG, bpG, ssG, psG, etc.	
D. k-times-removed	P^k		
D. in dim'n 4		K	
D. in dim'n 3	H	J, pK, bK	
D. in dim'n 2	t^jS, t^jO	tJ, tK, t^jG, bJ, pJ, bpK, pH, bH	

Table 1. This table lists all the generalized deltatopes that we have constructed. Some come from the Wythoff construction; others are pyramids or bipyramids, or are obtained by vertex shaving. The first column gives those that exist in every dimension; the next column gives those that only exist through dimension eight. Finally, we list those that only exist through dimension four.

We will prove later that most of the entries in this table are complete. However, we will only attempt a full classification for polytopes deltahedral in dimension at least three. Although Section 8 reviews the classification of deltahedra, we will not investigate how they can be put together to produce polytopes in higher dimensions, and will not look for polytopes deltahedral in dimension two beyond those in Table 1.

5. Polytopes with regular faces

In passing, we can mention further generalizations. Although any polytope with equal edge lengths could be considered deltahedral in dimension one, to classify analogs of deltatopes once-removed in $\mathbb{R}^3$, it might be interesting to put further restrictions. For instance, Zalgaller [Zal] showed that Johnson's list [Joh] of convex polyhedra with regular faces was complete: beyond the prisms and antiprisms, there are 92 examples. This list has been useful in the classification [KS2] of singularities of soap films with enforced symmetry (films in orbifolds). This list includes of course the square pyramid P_3, and the triangular prism G_3. However, we might want to allow as faces also the rhombus B_2; this would add more polyhedra to the list, like an octahedron with tetrahedra attached to one face or to two opposite faces. We could also view O_1 as an edge $\sqrt{2}$ times as long as the edge S_1. This makes P_2 and G_2

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be the two possible triangles combining these two edge lengths, as they should be. One could consider building polyhedra with these faces, which might be considered deltahedral in dimension zero in some standard way.

6. Spherical Duals

We will examine polytopes with faces of specific shapes by looking at their duals, which have specific structure at their vertices. If D is a polytope in $\mathbb{R}^n$, $n = m+1$, then any m-face of D has a unique (outward, unit) normal vector in S^m. These points are the vertices of a complex or net in S^m, called the spherical dual D^* of the polytope D. In fact, the rest of this complex D^* is given in a similar manner by the Gauß map, which takes any point x on D to the set in S^m consisting of normals to all supporting hyperplanes at x. If x is in the interior of a k-face of D, then its Gauß image is a subset of the equatorial (or, totally geodesic) S^{m-k} which is orthogonal to the k-space of the face. The Gauß image of the polytope fills out the entire sphere, with the vertices of D mapping to m-dimensional cells of D^*.

Suppose X is a face of D and has normal x. The point $x \in S^m$ is a vertex of the dual D^*, and we will look at its *link* in this dual net, in other words the intersection of D^* with a small S^{m-1} around x. This link will be the spherical dual X^* of the original face X. (X is a polytope in $\mathbb{R}^m$, so its dual is a net in S^{m-1}.)

The dihedral angles of D and of its faces are related to the edge lengths and vertex angles of its dual complex. Suppose X and Y are two adjacent faces of D, meeting along an $(m-1)$-face Z. The exterior *dihedral angle* between X and Y is the angle between their normal vectors; this clearly equals the (angular) length of the edge dual to Z, which lies in the unit S^1 perpendicular to Z. (The interior dihedral angle is the supplement of the exterior dihedral angle.) Applying this to X and its dual (the link of x as in the previous paragraph), we find that the (interior) angles at the vertex x of the net dual to D equal the corresponding (exterior) dihedral angles of X; this will be of central importance in our classifications.

Shrinking an edge of the dual until its two ends meet corresponds to opening out the corresponding two faces of D until they are in a common hyperplane. When this happens, we can merge them into a single face of a new type, and merge the dual vertices. But occasionally we will find it convenient to still consider these as separate faces, with the polytope no longer *strictly* convex.

The dual of the vertex figure of D at a vertex v is combinatorially equivalent to the cell corresponding to v in the dual of D. (This cell must of course be distorted to make it lie in a single sphere.)

7. Spherical Trigonometry

We will be considering deltatopes, polytopes whose faces are all regular simplices. Suppose D is a deltatope, and consider the 2-skeleton of its dual net. By the results of the previous section, we know each vertex angle in this net has cosine

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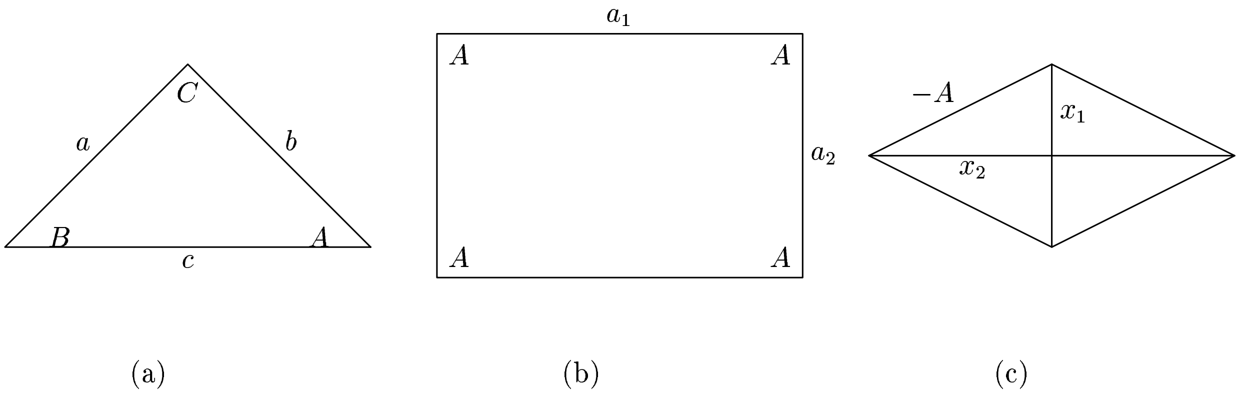


Figure 1. The spherical triangle (a) has edges with cosines a, b, and c, and sides with cosines A, B, and C, which satisfy the laws of sines and cosines. The quadrilateral (b) has angles A and sides a_1 and a_2. The polar quadrilateral (c) has sides $-A$.

$\hat{S}_m$. The 2-faces are thus polygons with equal angles in some S^2. Considerations like this lead us to study some of the relations of spherical trigonometry. We will always specify angles and (angular) lengths in terms of their cosines. Note that the negation of cosines corresponds to supplementation of angles, and that order relations are reversed for cosines.

Suppose we have a spherical triangle whose sides have cosines a, b, and c, and whose interior angles have cosines A, B, C, as in Figure 1(a). We always assume that our polygons have sides and angles between 0 and π, so that the sines of these sides and angles are positive. Then the law of sines can be expressed as

$$\frac{1-a^2}{1-A^2} = \frac{1-b^2}{1-B^2} = \frac{1-c^2}{1-C^2},$$

and the law of cosines says that

$$c - ab = C\sqrt{1-a^2}\sqrt{1-b^2},$$

where here and henceforth all square roots are positive. Note in particular that a right triangle (with $C = 0$) has sides with $a = A/\sqrt{1-B^2}$, $b = B/\sqrt{1-A^2}$, and $c = ab$. Also, an isoceles triangle with $A = B$ has base and sides with, respectively,

$$c = (C + A^2)/(1 - A^2), \qquad a = \frac{A}{\sqrt{1-A^2}}\sqrt{\frac{1+C}{1-C}}.$$

In particular, an equilateral triangle has $a = A/(1 - A)$.

It is also useful to consider polarity. Dual to each edge of a triangle is a vertex at the pole of the great circle the edge lies in. (We could specify which pole by orienting edges, but this will not be important.) These vertices are connected by great circle arcs for which the original vertices are poles. It is easy to see that the side lengths of the polar (or dual) triangle have cosines $-A$, $-B$, and $-C$, while its interior angles have cosines $-a$, $-b$, and $-c$. Applying the law of cosines to this triangle gives the dual law of cosines for the original triangle:

$$C + AB = c\sqrt{(1-A^2)(1-B^2)}.$$

This allows us to compute any side of a triangle given its angles. (Of course, here, as for the usual law of cosines, we could permute A, B, C in order to find other sides.)

A square (meaning a quadrilateral with equal angles and equal sides) whose angles have cosine A can be divided into eight right triangles, and we can check that its sides will have cosine $(1+A)/(1-A)$.

For more general polygons whose angles are known, it is often useful to consider polarity again: we get a polygon with known side lengths, which seems intuitively easier to handle. In particular, the useful operation of drawing diagonals in this polar polygon corresponds to the less obvious process of extending (nonadjacent) sides of the original polygon until they meet.

Consider now an equiangular quadrilateral with angles given by A (the cosine, as always). Its opposite sides have equal length, so there are two (cosines of) side lengths, a_1 and a_2. We know that a square has $a_1 = a_2 = A/(1-A)$. But all other lengths are possible, with limiting cases when $a_1 = -a_2 = \pm 1$. To find the relation between a_1 and a_2, draw the two diagonals of the polar rhombus (of side $-A$), which cut it into four right triangles of sides x_1 and x_2, say. Then by our formulas for right triangles,

$$x_1 x_2 = -A, \qquad \text{while} \quad a_i = (A^2 + 1 - 2x_i^2)/(1-A^2).$$

If we set $B = 1 - A^2$, this can be written as

$$(Ba_1 + B - 2)(Ba_2 + B - 2) = 4(1-B),$$

(The special case $B = 3/4$, appropriate for deltahedra, appears in [Tay].)

If we specify any four fixed angles for a quadrilateral, there will be a one-parameter family of possible shapes. Specifying one side length determines the shape. To maximize one side length, we let (at least) one of the others shrink to a point; then the quadrilateral degenerates into a triangle. (In the polar picture, we have a quadrilateral with specified sides. To minimize one interior angle, we must let another increase to a half-turn; again we end up with a triangle. To check this, draw a diagonal and minimize its length.)

Similarly, there is a two-parameter family of pentagons of fixed angles. To maximize a side length given these angles, we shrink two other sides to points. In particular, the maximum side length for a pentagon with five equal angles C is the section "Strictly Convex Deltahedra"

sec:hedra

long side of a triangle with one angle C and two angles given by $A := 1 - 2C^2$ (the cosine of the supplement of the double angle). By our formula for isosceles triangles, the long side is given by $c = (C + A^2)/(1 - A^2)$. If $C = S_2 = -1/2$, then $c = -1/3$ (and the other sides have $a = 1/3$), while if $C = S_3 = -1/3$, then $c = 11/16$. For pentagons with equal angles of one-third turn ($C = -1/2$), Taylor gives a formula relating adjacent edges a and b to the opposite edge c, which can be expressed as:

$$2c = \frac{1}{3} + a + b + ab - 2\sqrt{1-a^2}\sqrt{1-b^2}.$$

We will not need this formula here, except to note that it implies among pentagons with two equal adjacent sides, the longest those can be is $1/3$, in the degenerate case just described. Here we should use Cauchy instead.

8. Strictly Convex Deltahedra

A *deltahedron* is a polyhedron in $\mathbb{R}^3$ whose faces are all equilateral triangles. It is interesting in other contexts to study nonconvex examples. For instance Cervone [Cer] is interested in finding the smallest example which is topologically a torus (one is known with 36 faces: a union of three octahedra and nine tetrahedra, with three-fold symmetry); he has also constructed nonorientable surfaces as immersed deltahedra. The (nonconvex) deltahedra which are topologically spherical and have only five- and six-fold vertices seem to be related to configurations of certain organic molecules [Cas] and to configurations of point charges on a sphere [KS1]. But here we are interested only in convex deltahedra.

In fact, since exactly six equilateral triangles fit around a point ($S_2 = -1/2$), there is an important distinction between strictly convex deltahedra (where each face in the sense of our definition of convex polytope is a triangle) and other convex polyhedra made of equilateral triangles, but with some pairs of triangles in the same plane. (This distinction is not important in higher dimensions.) For instance, a truncated tetrahedron can be made from triangles, but is not a strictly convex deltahedron. We could also subdivide the faces of any deltahedron into smaller triangles, and thus get an infinite number of examples, while losing strict convexity.

We will see that there are just eight strictly convex deltahedra, with 4, 6, 8, 10, 12, 14, 16, and 20 faces. Evidently this classification has been rediscovered several times. Considering the soap-film problem, Lamarle [Lam] found most of the deltahedra (as the dual spherical nets) in 1864, but he missed one. In 1947, Freudenthal and van der Waerden [FvdW] classified the deltahedra as the only convex polyhedra with congruent regular faces, aside from the cube and dodecahedron. In 1964, Heppes [Hep] classified isogonal spherical nets, evidently unaware of the 1947 work; it was his classification that was cited by Taylor [Tay] in 1976 in her study of soap films. In the early 1960s, Johnson [Joh] listed all the convex polyhedra with regular faces (there are 92 besides the prisms and antiprisms) and Zalgaller [Zal] proved the

% sec:sphtrig
% fig:squares

list complete. We review the classification of deltahedra in this section because the techniques will be useful later.

Let D be a (strictly convex) deltahedron. Its dual D^* is a net in S^2 with all edges meeting in threes at equal angles. The faces of this net have 3, 4, or 5 sides, depending on the degree of the corresponding vertex of D. If the numbers of such vertices are n_3, n_4 and n_5, then Euler's relation gives $3n_3 + 2n_4 + n_5 = 12$. Note that a vertex of degree three is rigid (the dual triangle in S^2, with fixed angles, is determined).

Also, a degree-three vertex cannot be adjacent to a degree-five vertex without losing strict convexity, although we can get a degenerate case. This can be seen easily from a physical model: for instance an octahedron with one adjacent tetrahedron can be made from ten triangles. But three pairs of triangles are in common planes. If we keep just the faces around the three-fold and one five-fold vertex, the five-fold vertex is flexible, but as soon as we make it convex on one side, it loses convexity on the other side. To see this more rigorously, we recall our computations for equiangular spherical pentagons. Since the angles of our polygons are given by $S_2 = -1/2$, we found the side of a triangle is $-1/3$, while this is also the longest side of a pentagon. (As in Section 7, we specify all angles and lengths in terms of their cosines.) A nondegenerate pentagon thus cannot have a side long enough to be shared with a triangle.

So suppose first that our deltahedron D has a three-fold vertex. If there is an adjacent vertex also of degree three, then we have two adjacent triangles in the net D^*. The two other polygons adjacent to both of these have two sides $-1/3$, so they are also triangles, and we have a tetrahedron. Otherwise, the triangle is surrounded by three quadrilaterals of sides $-1/3$ and $7/9$. The opposite edges bound another triangle. The dual cell is a triangular prism, and D is a triangular bipyramid, the union of two tetrahedra sharing a face.

Any other D has only four- and five-fold vertices, so its dual is made of quadrilaterals and pentagons. We find that Euler's formula now gives $E/3 = F/2 = V - 2 = 10 - n_4$, though this will not be needed in our classification.

Suppose first that there are three adjacent quadrilaterals. If the first has sides a_1 and a_2, so must the second, as in Figure 2. Then the third has all sides a_1, so it, and the first two, must be squares, with side $1/3$. Now, we can continue examining adjacent faces. Each has two adjacent edges $1/3$, which is too long for any nondegenerate pentagon, so it must be another square. Thus the dual is a cube, and D is an octahedron. Note that the fact that we could fit a degenerate pentagon next to three squares corresponds again to the fact that a tetrahedron adjacent to an octahedron shares some of its face planes.

The remaining cases can be analyzed merely in terms of the topology of trivalent planar graphs—we make no more use of the geometric conditions on angles, except to show that a deltahedron does exist for each graph.

% fig:twoquads
% fig:pents
% tab:deltahedra

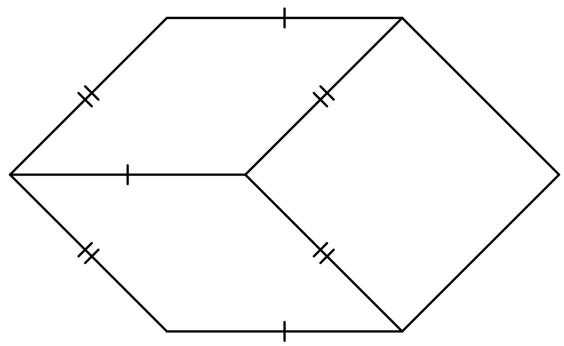


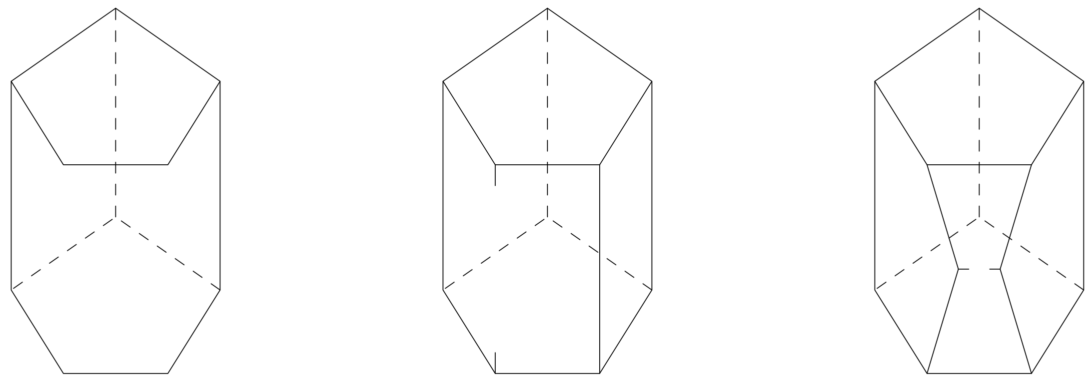
Figure 2. Any two adjacent quadrilaterals (of equal angles) will be congruent, so if a third is adjacent to both of them, it must be a square. Then since the other quadrilaterals have one side equal to that of a square, they are also squares.

If D is not an octahedron, but there are two adjacent quadrilaterals, we must have the situation in Figure 3. Either there is a third square in a row, or we have a pentagon at either end. In each case we end up with a hexagonal region with two free ends entering; these must be connected to each other to avoid regions with more than five sides. The first case gives a pentagonal prism, meaning D is a pentagonal bipyramid. The second case gives a deltahedron with 12 faces and four-fold symmetry group.

If D has only five-fold vertices, then it is the icosahedron. Otherwise, draw its dual as a planar trivalent graph with a quadrilateral on the outside, and four adjacent pentagons (see Figure 4). If we put another quadrilateral adjacent to a pair of these, we are forced into one pattern; if we put pentagons in all four places, another pattern is forced. These two graphs correspond to deltahedra with 14 and 16 faces. The 14-hedron is a triangular prism with the squares replaced by pyramids; the 16-hedron is a square antiprism with the squares replaced by pyramids.

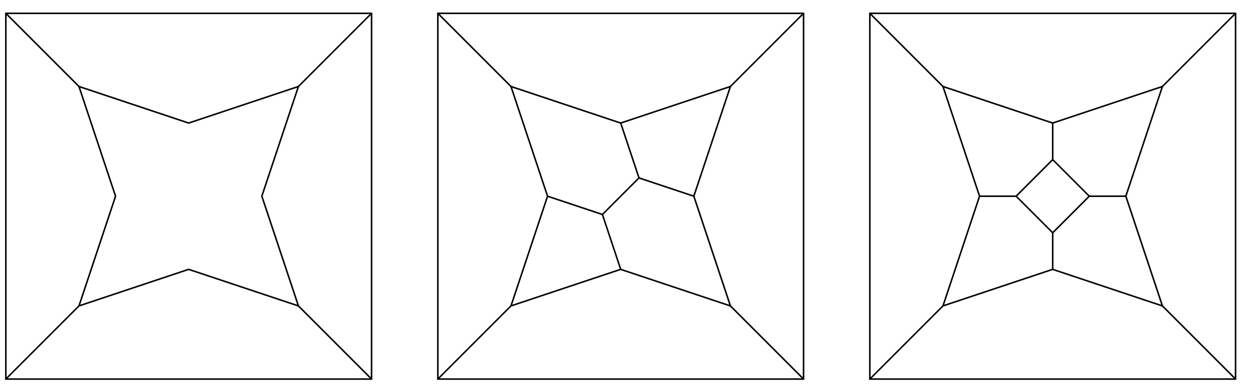
This completes the review of the classification. There is a unique deltahedron for each even number of faces between 4 and 20 inclusive, except 18. We have constructed all but the 12-hedron, which will be exhibited below.

Since each deltahedron has some symmetry, there is a nice way to orient it with respect to rectangular coordinates. We give in Table 2 the deltahedra, with



(a) (b) (c)

Figure 3. If there are two adjacent quadrilaterals, and a pentagon somewhere, we must have situation (a); if we put a third quadrilateral we then get (b); otherwise we must have (c). In either case, the two loose ends must be joined directly.



(a) (b) (c)

Figure 4. If we have a quadrilateral not adjacent to any others (a), we can put a quadrilateral across two of the pentagons (b), or use only pentagons (c). In either case the rest of the diagram is forced.

section "Convex Deltatopes in Higher Dimensions" sec:delta

the normal vectors to each face (which, when normalized, are the vertices of the dual net in S^2). The dihedral angles can of course be found by taking dot products of the normals. This specific information does not seem to be recorded elsewhere. ([Tay] for instance lists dihedral angles, but in some cases these are merely numerical approximations.) It would be of use in classifying generalized deltatopes.

The normals for the regular polyhedra are well-known. For the k-gonal bipyramids, we find the normals by taking k of the normals from the corresponding regular polyhedron, and reflecting them in the direction of their average. The deltahedra with 14 and 16 faces can be constructed from a triangular prism and square antiprism, respectively, by fitting pyramids on all the square faces. (Thus the gyroelongated square bipyramid might also be called the biaugmented square antiprism.) The snub disphenoid is the hardest to construct, since it only has eight-fold symmetry. To find its normals, we note that by symmetry, they can be taken to be $(\pm1, 0, \sqrt{\alpha})$, $(0, \pm1, -\sqrt{\alpha})$, $(\pm1, \pm\sqrt{\beta}, \sqrt{\gamma})$, $(\pm\sqrt{\beta}, \pm1, -\sqrt{\gamma})$. Then we must impose the condition that each face is equilateral. This means that if n is the normal to one face, and n_i $(i = 1, 2, 3)$ are the normals to the adjacent faces, then the angle between $n \times n_i$ and $n \times n_{i+1}$ must have cosine $-\frac{1}{2}$. This allows us to solve for α, β and γ; we find that

$$\alpha = \beta + \gamma, \quad \gamma = \alpha \left(\frac{2-\alpha}{4+\alpha}\right)^2, \quad 2\alpha^3 + 39\alpha^2 = 64,$$

and in fact α is the unique positive root of this last equation.

9. Convex Deltatopes in Higher Dimensions

We now define a *deltatope* in $\mathbb{R}^n$, $n > 2$ to be a (convex) polytope whose faces are all regular m-simplexes. (Here as always, we set $m := n - 1$.) We have seen that in $\mathbb{R}^3$ there are eight deltatopes. In any dimension, it is easy to find three deltatopes: the simplex and orthoplex, and the simplicial bipyramid. The simplex of course has $n + 1$ faces and the orthoplex 2^n. The simplicial bipyramid can be obtained as the union of a simplex with its reflection across a face, and has $2n$ faces.

For $n > 4$, we will see that these are in fact the only possibilities, but for $n = 4$, there are two others, the tetraplex and the icosahedral bipyramid.

The main tool for the classification is again the dual, a net in the sphere. We look at first the 2-cells in the dual, then the 3-cells, and so on, which corresponds to looking in the deltatope at the structure around $(m - 1)$-faces, then around $(m - 2)$-faces, and so forth.

So let D be a deltatope in $\mathbb{R}^n$, where $n \geq 4$. Each vertex in the dual of D corresponds to a face of D, which is an m-simplex. Thus the edges out of this vertex are in a simplicial pattern, and, in particular, are at angles with cosine $S_m = -1/m$. Thus any 2-cell in the dual is a equiangular spherical k-gon, with $k = 3$, 4 or 5, and with $k = 5$ possible only when $n = 4$.

% sec:sphtrig

Tetrahedron	4	$(-1,-1,-1)$, cyc$(-1,1,1)$
Triangular bipyramid	6	cyc$(-3,3,3)$, cyc$(-5,1,1)$
Octahedron	8	$(\pm1,\pm1,\pm1)$
Pentagonal bipyramid	10	$(5,5,\pm5)$, $(0,5\tau,\pm5/\tau)$, $(5\tau,5/\tau,0)$, $(1-2\tau,3-6\tau,\pm5)$, $(-4-2\tau,-2-\tau,\pm5/\tau)$, $(3\tau-4,-\tau-7,0)$
Snub disphenoid	12	$(\pm1,0,\sqrt{\alpha})$, $(0,\pm1,-\sqrt{\alpha})$, $(\pm1,\pm\sqrt{\beta},\sqrt{\gamma})$, $(\pm\sqrt{\beta},\pm1,-\sqrt{\gamma})$
Triaugmented triangular prism	14	cyc$(0,-3,-3)$, cyc$(4,1,1)$, perm$(2,\sqrt{6}-1,-\sqrt{6}-1)$, $\pm(\sqrt{6},\sqrt{6},\sqrt{6})$
Gyroelongated square bipyramid	16	$(\pm\sqrt{2},0,1)$, $(0,\pm\sqrt{2},1)$, $(\pm1,\pm1,-1)$, $(\pm2,0,\nu)$, $(0,\pm2,\nu)$, $(\pm1,\pm1,-\nu)$,
Icosahedron	20	$(\pm1,\pm1,\pm1)$, cyc$(0,\pm\tau,\pm(\tau-1))$

Table 2. This table gives the name (from [Joh]), number of faces, and normal vectors to these faces, for each of the eight strictly convex deltahedra. We write cyc(x,y,z) to abbreviate the three cyclic permutations of the vector (x,y,z), perm(x,y,z) for all permutations, τ for the golden ratio $(1+\sqrt{5})/2$, and ν for the quantity $2^{1/4}-2^{-1/4}$. The coordinates for the snub disphenoid satisfy the cubic equations given in the text, so that $\alpha\approx1.242077$, $\beta\approx1.216122$, $\gamma\approx0.025965$. For each polyhedron but the one with 16 faces, all the normals shown have the same length; however we have not normalized this length to be 1.

First assume we do have a pentagonal face (with $n=4$, hence angles $-1/3$). The longest possible side of this pentagon has cosine $11/16$, as was shown in Section 7. This is shorter than the side of a triangle, $-1/4$, and even shorter than the side of a square, $1/2$. So suppose somewhere we do have a pentagon adjacent to a quadrilateral. In the dual to D, there is a 3-cell C that they both bound. The boundary of C is polygonal with three-fold vertices. The two common neighbors of our pentagon and quadrilateral in the boundary of C, we claim, must each be quadrilaterals. The side a neighbor shares with the pentagon is too short to belong to a triangle, and the side shares with the quadrilateral, being the long side of the quadrilateral, is too long to belong to a pentagon. Continuing in this way, we get congruent quadrilaterals (of sides $11/16$ and thus $1/4$) ringing the pentagon, and must close the 3-cell C with another pentagon. Clearly the pentagons are regular, and C is a pentagonal prism. The 3-cell adjoining C across a quadrilateral face Q must be of the same type, since adjacent to long sides of Q there can only be congruent quadrilaterals. Continuing in this way, we find the whole structue of the dual to D; it is a dodecahedral prism, and D is an icosahedral bipyramid.

% fig:squares
section "Generalized Deltatopes"
sec:general

If, on the other hand, the dual to D has a pentagonal face, but no pentagon adjacent to a quadrilateral, then all faces are pentagonal. Each 3-cell in the dual is then a dodecahedron, meaning that the link of each vertex in D is an icosahedron. Dodecahedra meeting three around an edge form a dodecaplex (or 120-cell, see [Sul] or [Cox]) and D must be the dual tetraplex (or 600-cell) I_4. mention Cauchy rigidity

Thus we see that in $\mathbb{R}^4$ there are two deltatopes, the tetraplex I_4 and the icosahedral bipyramid bI_3, which somewhere have an edge with five tetrahedra around it. Other deltatopes in $\mathbb{R}^4$, and all deltatopes in higher dimensions, have only triangles and quadrilaterals in their duals. We will now show this leads to only three possibilities.

Assume we have a deltatope D in $\mathbb{R}^n$, whose dual has only triangles and quadrilaterals. Remember that these polygons have angles $\hat{S}_m = -1/m$ and thus the triangle has side $\hat{S}_n = -1/n$, while a square would have side $\hat{O}_n = \frac{n-2}{n}$. Look at a three-cell C of the dual; its faces meet in threes. If it has n_3 triangles and n_4 quadrilaterals, Euler's relation shows that $3n_3 + 2n_4 = 12$, which makes it easy to see that the only combinatorial possibilities are the cube, tetrahedron, and triangular prism. In fact with our geometric conditions each of these is rigid. Triangles have fixed side length, so an adjacent quadrilateral must have sides $-1/n$ and $1 - 2/n^2 =: \hat{B}_n$. If C has three adjacent quadrilaterals, as in the cube, we can see these must all be squares by using the argument illustrated in Figure 2 again.

Now look at how these cells can be used to build up the dual to D. Across a quadrilateral bounding a prism can only be another prism; repeating this argument we find that the 4-cell including these two prisms is itself a tetrahedral prism. Continuing by induction, the whole dual is a simplicial prism, and D is a simplicial bipyramid. Similarly, if there is a cubical 3-cell, the adjacent cells are all cubes, so the 4-cells are hypercubes, and the whole dual is a cube of the appropriate dimension, meaning that D is an orthoplex. If there are no prisms and no cubes as cells of the dual, all cells are tetrahedra, and inductively the whole dual is a simplex, as is then D.

This concludes the classification of deltatopes in all dimensions. We would like to note an important aspect of the argument. If we are assembling a polytope in R^n from its m-faces or "cells", then whenever we attach exactly three of these around a "hyper-edge" (an $(m-2)$-face), this configuration is rigid—spherical trigonometry determines the dihedral angles (edge lengths in the dual). If, instead, there are four cells around a hyper-edge, this is not rigid (quadrilaterals have one degree of freedom). But if this quadrilateral is next to a triangle, or next to two adjacent quadrilaterals, then we get rid of this degree of freedom, and recover rigidity.

10. Generalized Deltatopes

The scarcity of deltatopes in higher dimensions leads us to consider some generalizations. These generalizations will also be necessary to our analysis of soap-film

% sec:soap
% sec:hedra

singularities in Section 11. Remember that a polytope P in $\mathbb{R}^n$ whose k-faces are regular simplices is called *deltahedral in dimension* k, or, equivalently, a *deltatope* $(m-k)$*-times removed.*

We will not try to fully classify generalized deltatopes in higher dimensions, but will classify the deltatopes once-removed (except in $\mathbb{R}^4$), and all polytopes which are deltahedral in dimension eight.

In $\mathbb{R}^3$, of course, generalized deltatopes are just deltahedra, and were classified in Section 8. There are so many deltahedra that it seems hard to classify deltatopes once-removed in $\mathbb{R}^4$. But in all higher dimensions, we will now classify the deltatopes once-removed. For $n > 5$, these are polytopes whose faces are simplexes, simplicial bipyramids, and orthoplexes. For $n = 5$, we must allow also the tetraplex and the icosahedral bipyramid as cells, but these produce no further examples.

First, let D be a polytope in $\mathbb{R}^n$ $(n > 4)$ whose faces are simplexes, simplicial bipyramids, or orthoplexes. Remembering that a bipyramid is the union of two simplexes, we will divide any such face into its two simplexes, and consider them as separate faces. Of course they lie in the same $\mathbb{R}^m$, so we lose strict convexity.

The dual to D is a net in S^m. We mark its vertices S or O depending on whether they correspond to simplexes or orthoplexes. Corresponding to an original bipyramid is a pair of vertices, each marked S, joined by an edge of zero length. In any 3-cell of D^*, an S vertex has degree three, while an O vertex has degree four. At a vertex S, edges meet at angles given by $\hat{S}_m$, while at a vertex O the angle is instead $\hat{O}_m$.

A 2-cell of the dual will be a spherical polygon with these angles. Since $\hat{O}_m = 1 - 2/m \leq 1/2$, we cannot have a polygon with three Os. (For $m = 4$, exactly three orthoplexes fit around a "hyper-edge"—but this lies flat in an $\mathbb{R}^4$, so repeating this construction around every hyper-edge leads to the honeycomb $\{3, 4, 3, 3\}$ whose vertices form the lattice with integer coordinates of even sum [Cox].)

A polygon with two O vertices has room for at most one S vertex, and this SOO triangle in fact exists only for $m < 8$. For $m = 8$, assembling two orthoplexes and one simplex around each hyper-edge leads to the E_8 lattice. We will see that deltatopes arising from SOO triangles for lower n are in fact the Gosset polytopes [Gos] related to this lattice.

In any dimension, we can have a triangle with vertices OSS. We can succeed in fitting in one more simplex only when $m = 4$; in this case there are indeed quadrilaterals $SOSS$.

Thus the 2-cells of the dual to D can be of types SSS or $SSSS$ (already considered in classifying deltatopes) or type OSS; if $m < 8$ there is also SOO, and if $m = 4$, finally $SOSS$.

We now find the edge lengths for these polygons, to see how they might be assembled together. Remember that their angles are S_m and O_m, and that SSS has side $-1/n$ while $SSSS$ can have any side length, but a square has side $\hat{O}_n = \frac{n-1}{n+1}$.

% sec:sphtrig
% tab:examples

Using the formulas from Section 7, we find that OSS triangles have sides $\hat{P}_n := -1/\sqrt{n}$ (twice) and $\hat{O}_n$, while SOO triangles have sides $\frac{m-2}{2\sqrt{m+1}}$ (twice) and $m/4-1$. Here, of course, the only nonrigid polygon is the quadrilateral. Although in general we now allow SS edges to shrink to a point, we cannot allow this in an $SSSS$, since then other edges would grow to a full half-circle. This would give adjacent faces of D with opposite normals, so D would degenerate into a lower dimension.

For $m > 4$, we can now complete the classification of deltatopes once-removed. If D is such a polytope with no orthoplicial faces, it is one of the three deltatopes. If there is an orthoplex, we have a triangle SOO or OSS in the dual. Note that the OS edges in these two kinds of triangles have different lengths, so either kind of triangle is adjacent (across OS edges) only to congruent triangles. Thus around any O vertex we find only one type of triangle.

In the SOO case, our first orthoplex touches 2^m others, which again must be surrounded in the same way. We see that every 2-cell in the dual is an SOO triangle. The rigidity of these triangles shows that there can only be one polytope D of this type; in fact this is the Gosset polytope G_n.

In the OSS case, we have an orthoplex completely surrounded by simplexes. Clearly the remaining vertices of all these simplices coincide, and D is an orthoplicial prism P_n.

Thus for $m > 7$ there are exactly four deltatopes once-removed in $\mathbb{R}^n$: the three deltatopes and the orthoplicial prism. For $m = 5, 6, 7$, there are these four plus the Gosset polytope. We have thus shown that Table 1 gave a complete list of deltatopes and deltatopes once-removed, in dimensions $m > 4$. check this assertion

For $m = 4$, remember that we are not yet trying to classify all deltatopes once-removed, but only those with simplex, orthoplex, and bipyramid faces. If there is no $SOSS$ quadrilateral in the dual, the classification proceeds as above, and we find the five possibilities mentioned there. Let us then look at the geometry of this quadrilateral. Its angles are $S_4 = -1/4$ and $O_4 = 1/2$. To get limits on the lengths of the two types of edges SS and OS, we consider the extreme case where one of the SS edges shrinks to a point. We are left with a triangle with $A = 1/2$, $B = 7/8$, $C = -1/4$, from which we get $a = 3/5$, $b = 2/\sqrt{5}$, $c = 1/\sqrt{5}$. Remember that $3/5 = q_4$ is the side of a square, and $1/\sqrt{5}$ is the shorter of the two other possible SO edges (the one in the OSS triangle). Mention cauchy

This value b represents the shortest possible SO edge in such a quadrilateral, while c is the longest. Similarly, a is the longest possible SS edge, while such an edge can be arbitrarily short. Ignoring for now the possibility of a degenerate $SOSS$ quadrilateral, which would correspond to a bipyramid face of D, we can show there are no further deltatopes once-removed in $\mathbb{R}^5$ with only simplex and orthoplex faces: only the five ones with analogs in higher dimensions exist.

The longest SO edge of a nondegenerate $SOSS$ is shorter than any other possible SO edge, so the adjacent faces are again $SOSS$ quadrilaterals. In a 3-cell of the

% fig:soss

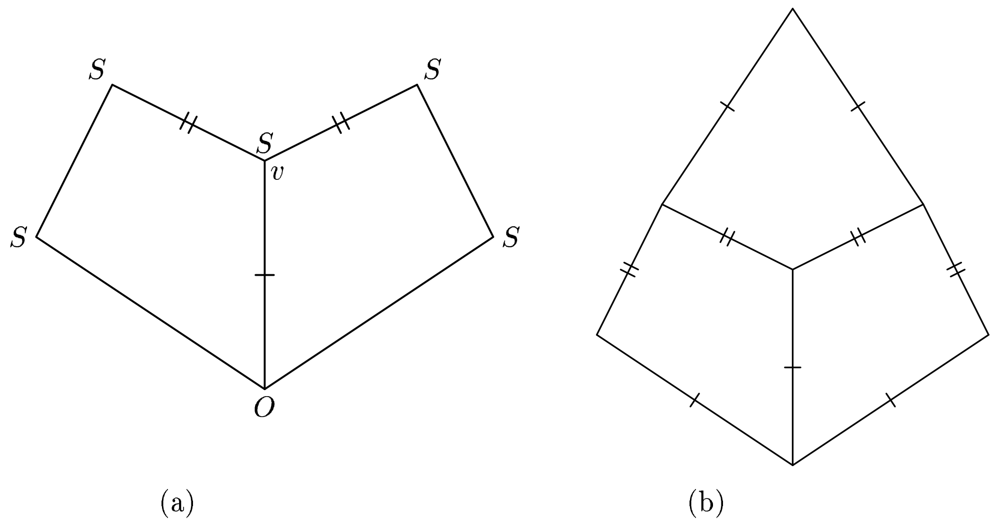


Figure 5. Two adjacent nondegenerate $SOSS$ quadrilaterals (a) must in fact be congruent, since each has only a single degree of freedom. Since the edges with double marks are too short to belong to a square, the adjacent 2-cell at v must be another congruent $SOSS$ as shown in (b). Therefore those edges are exactly the right length to belong to a symmetric $SOSS$, so all three are congruent.

dual, look at the S vertex v shared by two of these quadrilaterals (see Figure 5). It has two equal SS edges, but these are too short to belong to a square $SSSS$, so the third 2-cell is again $SOSS$. Then all these $SOSS$ quadrilaterals are symmetric. Continuing, we find all adjacent 2-cells are symmetric $SOSS$ quadrilaterals, and the 3-cell is bounded by eight of them. (It is the dual of a square antiprism.) Then all adjacent 3-cells have to have the same type, and we would have a polytope made by always attaching three simplexes and one orthoplex around a hyper-edge. Such a polytope (the square antiprism) exists in $\mathbb{R}^3$, but cannot exist in higher dimensions. If there were one in $\mathbb{R}^5$, its vertex figure would be one in $\mathbb{R}^4$, and so on. In $\mathbb{R}^4$, it is possible to arrange two octahedra and eight tetrahedra around a vertex in an antiprism fashion. But now consider the eight exposed faces of the tetrahedra. We cannot put an octahedron across one of these faces, for then one of its edges would be shared with the first octahedron (but the edges are supposed to be $SOSS$). But if we put tetrahedra on all eight, then there are some edges with four tetrahedra already.

rewrite the rest of this par. Talk somewhere about construction rules

In $\mathbb{R}^5$ there is in fact one more deltatope once-removed Q_5, constructed from degenerate $SOSS$ quadrilaterals. It has 17 vertices, which can be taken to be the ten permutations of $(\pm 2, 0, 0, 0, 0)$ except $(-2, 0, 0, 0, 0)$, together with the eight points $(-2, \pm 1, \pm 1, \pm 1, \pm 1)$ with an even number of $+$ signs. This can be constructed by

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assembling sixteen simplexes around $(2, 0, 0, 0, 0)$ as if constructing O_5, but then extending half of them to bipyramids B_4. The opposite vertices of these bipyramids are then the eight vertices of an orthoplex O_4 centered around $(-2, 0, 0, 0, 0)$. Across each face of this orthoplex fits a simplex; across the faces of the original simplexes go orthoplexes. We end up with sixteen simplexes, nine orthoplexes, and the eight bipyramids, for a total of thiry-three faces, Of course, Brakke's comparison surface for the bipyramid shows this cannot give a soap-film singularity candidate.

The orthoplicial pyramid cannot be used to construct any further deltatopes twice-removed for $n > 8$, except the family P_n^k. (The only possibility would be $PPSS$ quadrilaterals.) Thus we have completely classified all polytopes deltahedral in dimension seven. expand

11. Relating Deltatopes to Soap-Bubble Singularities

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This proceeds by induction, but in higher dimensions we must allow certain generalized deltatopes. Steiner trees show that Taylor only had to look at deltahedra (and in general we only have to look at some subset of generalized deltatopes). Her result shows that in $\mathbb{R}^4$ we need only look at deltatopes (and in $\mathbb{R}^5$ only at deltatopes once-removed, etc). Brakke's result in $\mathbb{R}^4$ shows that in fact in $\mathbb{R}^5$ we need only look at deltatopes once-removed made of S and O. We classified these above: simplex, orthoplex, simplicial bipyramid, orthoplicial pyramid, Gosset (here the hemi-cube). The first three are calibrated by Brakke. The orthoplicial pyramids P_n^k are calibrated for all cases $n - k > 3$ by a modification of the Brakke hypercube calibrations. Thus in $\mathbb{R}^n$, $(n > 4)$ we know there are at least $n - 1$ different polyhedral singularities in soap films.

We do not have a full list of candidates for singularities in all dimensions, but it will be interesting to investigate the new candidates related to the Gosset polytopes; if these cones are not minimizing, our list of candidates is complete.

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