# Wienholtz's Proof of a Conjecture about Projections of Small Diameter

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In 1997, Rob Kusner and I conjectured that any space curve of length $L$ has a projection of diameter at most $L/\pi$. In late 1999, Daniel Wienholtz of Bonn proved a slightly stronger result, in a series of three preprints. This page summarizes his argument.

**Proposition.** *Given any closed curve $\Gamma$ in $\mathbb{R}^n$, there are two antiparallel bitangent support planes for $\Gamma$, with their points of tangency interleaved.*

*Proof.* Suppose not. Then for any unit vector $v$, we can divide the circle parameterizing $\Gamma$ into two complementary arcs $\alpha(v)$, $\beta(v)$, such that the maximum of the height function in direction $v$ is achieved only (strictly) within $\alpha$, and the minimum is achieved only (strictly) within $\beta$. In fact, these arcs can be chosen to depend continuously on $v$. Now let $a(v)$ in $\mathbb{S}^1$ be the midpoint of $\alpha(v)$. Consider the continuous map $a\colon \mathbb{S}^{n-1} \to \mathbb{S}^1 \subset \mathbb{S}^{n-2}$. By Borsuk-Ulam, there must be some $v$ such that $a(v) = a(-v)$. But the height functions for $v$ and $-v$ are negatives of each other, so maxima in direction $v$ live in $\alpha(v)$ and in $\beta(-v)$, while minima live in $\beta(v)$ and $\alpha(-v)$. This is impossible if $a(v)$ is the midpoint of both $\alpha(v)$ and $\alpha(-v)$. (In fact, a sensible choice for $a$ makes it antipodally equivariant: $a(-v) = -a(v)$.) $\square$

**Lemma.** *If $\Gamma$ is a curve in $\mathbb{R}^{n+m}$ of length $L$, and its projections to $\mathbb{R}^n$ and $\mathbb{R}^m$ have lengths $a$ and $b$, then $a^2 + b^2 \le L^2$.*

*Proof.* It is enough to prove this for polygonal curves and then pass to an appropriate limit. Each edge projects to edges of length $a_i$, $b_i$. Consider the polygon in $\mathbb{R}^2$ with successive side vectors $(a_i, b_i)$. Its total length is $L = \sum \sqrt{a_i^2 + b_i^2}$, but the distance between its endpoints is $\sqrt{a^2 + b^2}$. $\square$

**Theorem.** *Any closed curve $\Gamma$ in $\mathbb{R}^n$ of length $L$ lies in some cylinder of radius $L/2\pi$.*

*Proof.* The case $n = 2$ follows directly from Cauchy-Crofton, since $\Gamma$ has width at most $L/\pi$ in some direction. We prove the general case by induction. So given $\Gamma$ in $\mathbb{R}^n$, find two parallel interleaved-bitangent support planes with normal $v$, as in the first theorem. Let $d$ be the distance between these planes; the thickness of the slab in which $\Gamma$ lies. Project $\Gamma$ to a curve $\gamma$ in the plane orthogonal to $v$, and call its length $l$. By induction, $\gamma$ lies in a cylinder of radius $l/2\pi$. Clearly, $\Gamma$ lies in a parallel cylinder of radius $r$, centered in the middle of the slab, where $r^2 = (l/2\pi)^2 + (d/2)^2$. So we need to show that $(l/2\pi)^2 + (d/2)^2 \le (L/2\pi)^2$, i.e., $l^2 + \pi^2 d^2 \le L^2$. In fact, since $4d$ is the length of the projection of $\Gamma$ to the one-dimensional space in direction $v$, the lemma gives us $l^2 + 4^2 d^2 \le L^2$, which is better than we needed. $\square$

**Remark.** *Suppose the closed curve $\Gamma \subset \mathbb{R}^n$ has length $L$, and $p_1, p_2 \in \Gamma$ are points realizing its diameter. Then its projection $\gamma$ to the plane orthogonal to $p_1 - p_2$ has diameter at most $L/\sqrt{8}$.*

*Proof.* Let $a_1$, $a_2$ be preimages of a pair of points realizing the diameter $d$ of the projected curve $\gamma$. We may assume $\Gamma$ is a quadrilateral with vertices $a_1$, $p_1$, $a_2$, $p_2$; any other curve would be longer. (This reduces the problem to some affine $\mathbb{R}^3$ containing these four points. We can also reduce to $\mathbb{R}^2$: rotating $a_1$, $a_2$ independently about the line $l$ through $p_1$ and $p_2$ fixes the length, but maximizes the diameter $d$ when the points are coplanar, with $a_i$ on opposite sides of $l$.)

Along $\Gamma$, suppose the $a_i$ are interleaved with the $p_i$, so that the quadrilateral is $a_1 p_1 a_2 p_2$. Let $R$ be the reflection in the line $p_1 p_2$. Consider the vector $R(p_1 - a_1) + (a_1 - p_2) + (p_1 - a_2) + R(a_2 - p_2)$. Its length is at most $L$, but its component in each of two perpendicular directions is at least $2d$. Thus $L \ge 2\sqrt{2}d$.

Otherwise, the quadrilateral has two sides $a_1 a_2$ and $p_1 p_2$ each of length at least $d$ (and in perpendicular directions). The sum of these two side-vectors has length at least $d\sqrt{2}$, and since it equals the sum of the other two side vectors, their lengths add to at least $d\sqrt{2}$. Therefore $L \ge (2 + \sqrt{2})d$. $\square$