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Pressures in periodic foams

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A hitherto open problem in the geometry of foams asks: are the pressures in the bubbles of a (structurally) periodic foam in equilibrium necessarily periodic? We supply a proof that this is so. However, external forces can induce a pressure gradient even in periodic structures. We also establish that any equilibrium planar foam whose bubbles are congruent must have uniform pressure and be a fully periodic arrangement of hexagons.

Keywords: foams; bubbles; pressures; equilibrium structures; tilings; Kelvin foam

1. Introduction

It seems reasonable that if the *structure* of a foam in equilibrium is periodic, then the pressures must be periodic. But it appears that a proof of this assertion has not yet been published. Our theorem 2.1 provides a proof for foams in Euclidean space of arbitrary dimension, in the absence of external forces. However, we also give counter-examples to show that the statement fails when there are external forces, such as gravity, or when the ambient space is not flat. We then consider the structure of a planar foam consisting of congruent bubbles (i.e. bubbles which all have identical shapes). Theorem 3.1 shows that such a foam in equilibrium must consist of a fully periodic arrangement of hexagons.

Let us first of all state what we mean by a foam. In this context, we understand a foam to be a piecewise smooth arrangement of connected, bounded regions ('bubbles') tiling space (of arbitrary dimension). The bubble volumes are taken to be fixed. The bubble surfaces (or, more physically, the films constituting the boundaries between the bubbles) are subject to a surface energy, i.e. there is an energy cost proportional to the total area of the interfaces. In many models for foams this is the only energy involved. A foam *in equilibrium*, for which the first derivative of the surface energy is zero, then has a number of well-known characteristics. For instance,

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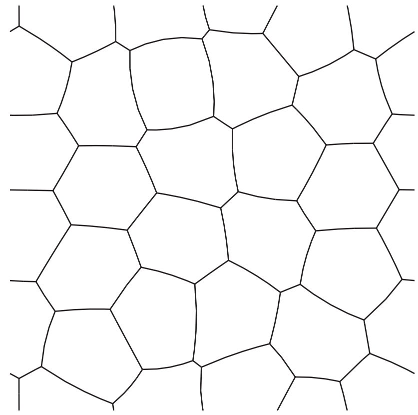

Figure 1. Example of a periodic foam in equilibrium (no external forces are applied).

the Plateau rules say that films meet in threes at angles of 120°, along edges known as *Plateau borders*. (A fourfold X junction is not in equilibrium in our sense, because breaking it apart into two threefold Y junctions reduces the area to first order.) An example of such a foam in two dimensions is shown in figure 1. Each bubble has a uniform pressure and the signed mean curvature of each film is proportional to the pressure difference. One consequence is the *curvature sum rule*: the sum of the signed mean curvatures of all surfaces intersecting a generic closed path must be zero. In general there may be other contributions to the energy, for example, due to applied forces, and such criteria may then require modification. We will return to the issue of external forces later.

2. Periodic foams

We shall use the language of three dimensions, but our analysis does not depend upon the dimension of the space. We consider a foam whose geometry is repeated periodically with lattice vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, thereby tiling space. We will show that the following theorem is true.

Theorem 2.1. *In the absence of external forces, the bubbles in a foam in equilibrium with periodic structure in Euclidean space must have periodic pressures.*

Proof. The condition of mechanical equilibrium requires that no net forces act anywhere in the system. This implies that the forces on any sub-volume must cancel, and in particular this must hold for the primitive (Bravais) unit cell of the lattice. We shall show formally that this requires pressures to be periodic.

Locally, the pressures must accord with the Laplace law. That is, the pressure difference between two neighbouring bubbles is given by

$$\Delta p = \frac{2\sigma}{r}, \tag{2.1}$$

where σ is the surface tension (energy per area) of the film separating the two bubbles, and r is its radius of mean curvature. Since r is determined by the local geometry, it is periodic, and so all such pressure *differences* are periodic.

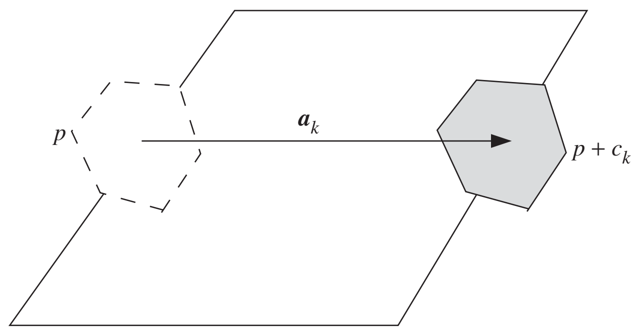

Figure 2. Equivalent bubbles on opposite sides of the primitive unit cell C for the lattice are related through translation by a lattice vector $\boldsymbol{a}_k$. Their pressures are related by an additive constant c_k, which we show must be zero.

Consider now some bubble B and its periodic copy shifted by some lattice vector $\boldsymbol{a}_k$. We denote their respective pressures as $p(B)$ and $p(B + \boldsymbol{a}_k)$, which may *a priori* be different. However, it follows from the Laplace condition that they can only differ by a constant independent of B,

$$p(B + \boldsymbol{a}_k) = p(B) + c_k, \tag{2.2}$$

where the constant c_k is the same for all bubbles, as illustrated in figure 2. To see this, select one bubble B_0 and choose c_k to satisfy (2.2). For any neighbouring bubble B', the Laplace condition then implies $p(B') - p(B_0) = p(B' + \boldsymbol{a}_k) - p(B_0 + \boldsymbol{a}_k)$, so (2.2) holds for B' as well, with the same c_k. Repeating by induction from neighbour to neighbour, (2.2) must hold for all bubbles B.

Our goal is now to prove $c_k = 0$. To do so, it is convenient to consider the primitive (Bravais) unit cell C of the lattice, a parallelepiped with edges along the vectors $\boldsymbol{a}_k$. The total net force on the contents of this cell is a sum of the forces on its boundary faces, due to pressure and surface tension. By periodicity, the surface tension terms make no net contribution, since the individual contributions cancel on opposite faces.

By the discussion above, the pressures on opposite faces of the parallelepiped C differ by the constants c_1, c_2, c_3. Hence, the total force due to these is

$$\boldsymbol{F} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + c_3\boldsymbol{b}_3, \tag{2.3}$$

where $\boldsymbol{b}_k$ are the reciprocal lattice vectors, normal to the faces of C. In three dimensions, these are given by $\boldsymbol{b}_k = \boldsymbol{a}_m \times \boldsymbol{a}_n$ with (k, m, n) being cyclic permutations of $(1, 2, 3)$. Note that $\boldsymbol{b}_k$ defines the (oriented) area of one pair of opposite faces of C.

Now, the contents of this primitive cell are entirely in local equilibrium, that is, they are in equilibrium under the total applied force. If there is no other *external* force to balance $\boldsymbol{F}$, then we must have $\boldsymbol{F} = \boldsymbol{0}$. Since the $\boldsymbol{b}_k$ are linearly independent, this means that $c_k = 0$. Hence, pressures of equivalent bubbles are identical, and therefore pressures are periodic. ■

Remark 2.2. The proof above depends on the fact that there are no external forces, and on the fact that the ambient space is flat, so that forces acting at different points can be added together in the usual way. The first part of our proof is true even without these assumptions, and we conclude more generally that there are constants

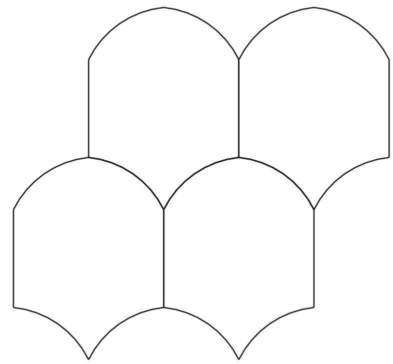

Figure 3. Example of a periodic foam 'loaded' by the weight of its Plateau borders. In this two-dimensional example, the Plateau borders are the isolated triple junctions, where three bubbles meet. The piece of the foam shown repeats by translation, and there is a vertical pressure gradient.

c_k leading to a uniform pressure *gradient*, in the sense that

$$p(B + n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + n_3\boldsymbol{a}_3) = p(B) + n_1c_1 + n_2c_2 + n_3c_3. \tag{2.4}$$

B. White (1995, personal communication) has suggested a more mathematically oriented proof for theorem 2.1. Periodic boundary conditions apply to the unit cell; identifying opposite sides gives a torus T. (Mathematically, this quotient torus is Euclidean space modulo the translational lattice, and thus carries a flat metric.) The periodic foam becomes a finite bubble cluster in T. The pressure gradient can be seen to equal the first variation of surface area under an infinitesimal translation of the cluster within T. But for a flat torus like T these are isometries and so this variation (and hence the pressure gradient) must be zero.

Real samples of liquid foam would not involve any periodic boundary condition. What then could be the relevance of the present results? Consider a foam which is *homogeneous* over length-scales much greater than the individual bubbles. The local average gas pressure is a well-defined quantity in this case. The arguments we give here suggest that this average pressure must be constant for a homogeneous foam in equilibrium without external forces. The average pressure can have a non-zero gradient when the foam is not in equilibrium, or when the structure is made inhomogeneous by a gradient of bubble size.

When external forces such as gravity are present, however, a non-zero pressure gradient can indeed be maintained in a periodic foam. In particular, our own work on drainage (Weaire *et al.* 2003) has led us to define the following loaded foam model. In a *loaded foam*, a uniform gravitational force (per unit length) acts downward on all of the Plateau borders, which contain most of the liquid in a physical foam. In this case, the total weight force per unit volume will be balanced by a pressure gradient of the kind discussed above. The effect upon the structure is to change the angles at which the films intersect, away from 120°. A simple two-dimensional example is shown in figure 3. We believe this model will be useful in the context of drainage, and particularly in relation to drainage instabilities. We will present such an analysis in a forthcoming paper (Weaire *et al.* 2004).

Similarly, periodic foams may have pressure gradients if the ambient geometry is not that of flat Euclidean space. The usual physical model for two-dimensional foams is bubbles confined between two closely spaced parallel glass plates. If the plates each have a periodic pattern of bumps (while remaining parallel), then this becomes a model for a foam in a two-dimensional curved space, with periodic curvature. A foam structure between these plates, even if it is periodic with the same periodicity as the bumps, could still support a pressure gradient. Here, the angles between films would still be equal to 120°, but the bumpiness of the ambient space allows the global geometry of the foam to differ from what we would expect in the Euclidean plane. Intuitively, the foam can support a pressure gradient, without simply moving in the direction of lower pressure, because films would have to stretch to go past the bumps. It seems reasonable to conjecture (as a referee suggested) that in any compact manifold with infinite fundamental group—other than a flat torus—there should be foams with pressure gradients.

For any curved ambient space, our above proof breaks down, because forces applied at different places can no longer be added; White's proof fails because a non-flat torus does not have infinitesimal translations in every direction. R. Kusner (2002, personal communication) has pointed out the connection between these: the infinitesimal translational symmetry of a flat torus leads via Noether's theorem to conserved quantities. These can naturally be viewed as forces, as for constant-mean-curvature surfaces in Korevaar *et al.* (1989), and they make rigorous the idea that forces applied at different points can be added when the ambient space is flat.

3. Planar congruent foams

By a *congruent* foam, we mean a foam in which any two bubbles are congruent. That is, the foam is a *monohedral* tiling by copies of a single bubble shape, the *prototile*. Here and in the following we use congruence in the general sense, allowing reflections (which we will call 'flips') of the prototile. In tiling theory, a monohedral tiling is called *isohedral* if, for any tiles T and T', there is a congruence of the whole tiling taking T to T'. That is, not only are T and T' congruent shapes, but they sit in the same way in the tiling. A special kind of isohedral tiling is one in which all these congruences are translations: the tiling is then said to be a *tiling by translations* of its prototile.

We will show that any planar congruent foam has equal-pressure bubbles and is necessarily a tiling by translations of a (straight-sided) hexagon.

The fact that the prototile has six (possibly curvilinear) sides follows from Euler's theorem, because the bubbles in an equilibrium foam meet in threes. Thus, we can describe the shape of any possible prototile by the sequence of (signed) curvatures $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ of its six edges. As shown in figure 4*a*, positive curvature represents a convex edge of the tile, and we record the six curvatures in anticlockwise order. Note that dihedral permutations (cyclic permutations or reversals) of a given sequence are equivalent for our purposes, since they represent the same underlying shape of the prototile (rotated or flipped). Note also that each edge fits against an edge of some congruent copy of the same prototile, so that for every edge of curvature $k > 0$ there is an edge of curvature $-k < 0$.

Similarly, we characterize a vertex v by the triplet of (signed) curvatures of the incident edges, as in figure 4*b*. Here, positive curvature refers to an edge curving right as it leaves the vertex, and we again list edges in anticlockwise order. Suppose we have a vertex v of type $\langle k_1, k_2, k_3\rangle$. By the curvature sum rule, we must have

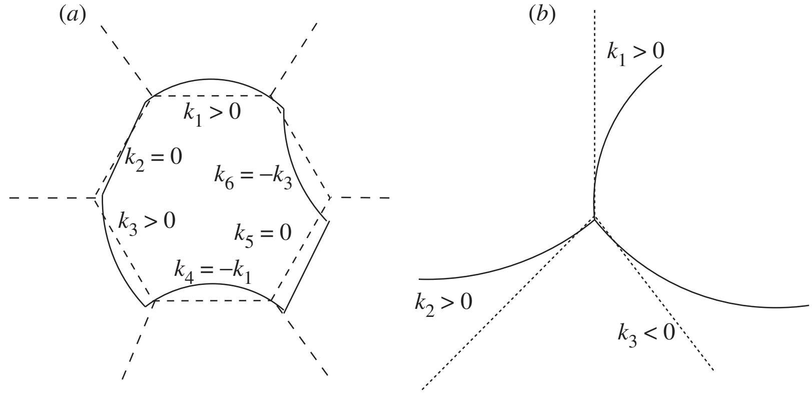

Figure 4. Illustration of the notation used (*a*) Tiles are of hexagonal topology, and they are parametrized by the sequence of the (signed) curvatures associated with their edges. The prototile shown has the sequence $\{k_1, 0, k_3, -k_1, 0, -k_3\}$. (*b*) Vertices have three incident edges, and can thus be characterized by three (signed) curvatures. Here the signs refer to a closed path in an anticlockwise direction around the vertex. The vertex shown has curvatures $\langle k_1, k_2, k_3\rangle$. It follows that the curvature pairs $(-k_1, k_2)$, $(-k_2, k_3)$ and $(-k_3, k_1)$, possibly reversed, must be present on the prototile.

$k_1 + k_2 + k_3 = 0$. Considering the three copies of the prototile meeting at v, we see that the prototile has consecutive edges of curvatures $(k_1, -k_3)$, perhaps in reverse order; similarly, there are consecutive edges of curvatures $(k_3, -k_2)$ and $(k_2, -k_1)$. Conversely, if (k_1, k_2) are two consecutive curvatures of the prototile, then the vertex between them has type $\langle k_1, -k_2, k_2 - k_1\rangle$, so the prototile must also have edges of curvature $\pm(k_2 - k_1)$.

Now consider again the curvatures at a vertex v. Flipping the foam if necessary to make at least two curvatures non-negative, we can write them as $\langle k, k', -(k + k')\rangle$ or $\langle k', k, -(k + k')\rangle$, with $0 \leqslant k \leqslant k'$.

One prototile consistent with this vertex has curvatures

$$\{k, -k', -(k + k'), -k, k', k + k'\};$$

we call this the *standard tile* T for v. Its opposite edges have matching curvatures, so the standard way to tile the plane with copies of T would be to match opposite sides (with no flips). We will show that, in fact, this *standard tiling* is the only possible one, and that it works only when all the curvatures vanish. Note that a standard tiling is isohedral, because the matching rule determines the tiling uniquely from any one tile.

Theorem 3.1. *A planar congruent foam in equilibrium (in the absence of external forces) is a doubly periodic tiling by translations of a (straight-sided, equiangular) hexagon; all bubbles have equal pressures.*

Proof. First let us show that a standard prototile T, tiling in the standard way, must have straight sides. Since the standard tiling is isohedral, this follows directly from lemma 4.1 below. However, here we prefer to present an elementary proof. Note that the rigid motion taking the edge of curvature $-k_1$ to that of curvature k_1 must be a translation: otherwise T and its copies under iteration of this motion would tile

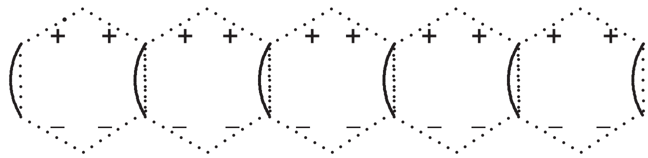

Figure 5. A geometrically impossible tile. Suppose tiles are required to line up along one direction for their edge curvatures to match. To tile the plane, this line must then be straight rather than curved. Consequently, no tiles can occur where all edge curvatures on one side of this line are positive, while all those on the other side are negative (as shown).

an annular strip instead of a straight strip, as in figure 5. But the edges on one side of this strip have curvatures $k_1 + k_2 \geqslant k_2 \geqslant 0$, while the edges on the other side have curvatures $-k_1 - k_2 \leqslant -k_2 \leqslant 0$. Because Plateau's laws require the angles at every vertex to be 120°, we must have the same total integrated curvature (sum of curvatures weighted by the edge lengths) on either side in order to make the strip straight. This can only happen when $k_1 = k_2 = 0$, meaning that T is a straight-sided equiangular hexagon, with opposite sides of equal length, tiling by translation.

Now, we will show that there can be no non-standard tilings. If v is any vertex of the prototile, we write its curvatures as above as k, k' and $-(k + k')$, where $0 \leqslant k \leqslant k'$. We want to show $0 = k = k'$, that is, to rule out all other possibilities.

(i) The generic case is $0 < k < k'$, and here the six curvatures $\pm k$, $\pm k'$, $\pm(k + k')$ are distinct, so any prototile T must use them once each. The enumeration of all possible cases is straightforward but tedious; it can be simplified by constructing an abstract graph with six nodes, each labelled by one of the six curvatures. Two nodes are connected by a link if the corresponding curvatures may occur on consecutive edges on a tile T, i.e. if their difference is again one of the six available curvatures. The resulting graph may have many links. However, the possible tiles T can be identified from this graph as those closed loops involving every node exactly once (known as 'Hamiltonian cycles'). We shall see that this unambiguously leads to the standard tile.

Let us first assume $k' \neq 2k$. Then the graph has only six links, forming a closed loop. Reading off curvatures in the corresponding order, we see that the resulting tiling is the standard tile $\{k, -k', -(k + k'), -k, k', k + k'\}$. In the special case $k' = 2k$, the curvatures available are $\pm k$, $\pm 2k$ and $\pm 3k$. Nine combinations of adjacent curvatures are admitted, so the resulting graph has nine links, as in figure 6. However, inspection shows that there is still only one Hamiltonian cycle (the closed path visiting each node exactly once), corresponding again to the same standard tile.

Having established that the tile T must be of the standard type, we need to consider how it can be used to tile space. Still assuming $0 < k < k'$, there is no choice about how to match edges to create a tile (if we matched the $\pm k$ edges *with* a flip, for instance, we would create an illegal vertex where the curvature sum rule would fail). Therefore, T can only tile in the standard way. But we have already seen that a standard tiling has equal pressures, which contradicts the assumption that the edges are curved.

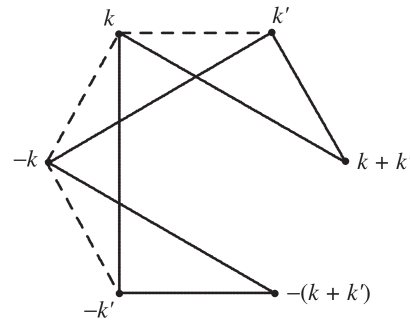

Figure 6. An abstract graph may be used to identify valid tiles. The six nodes are the available curvatures $\pm k$, $\pm k'$, $\pm(k+k')$, while the links connect those curvatures which may occur consecutively on a tile (without violating the curvature sum rule). Generically, there are six links (solid lines). When $k'=2k$, there are three additional links (dashed lines) but still the solid lines give the unique Hamiltonian cycle (closed loop visiting each node exactly once). It corresponds to the standard tile $\{k, -k', -(k+k'), -k, k', k+k'\}$.

(ii) Having reached a contradiction in case (i), we may now assume that there is no vertex v with distinct non-zero curvatures. That is, every vertex v must have edges with identical curvatures and/or straight edges.

So suppose now there is a vertex v with curvatures $\langle k, k, -2k\rangle$, for $k>0$. The six curvatures of the prototile T must include $\pm k$ and $\pm 2k$. We claim that $\pm k$ must be repeated. Otherwise, the edge of curvature $2k$ is consecutive on T with some edge of curvature $k' \neq k$; then either the vertex between them would have three distinct non-zero curvatures (which we have already excluded), or its type would be $\langle 0, 2k, -2k\rangle$. The latter case would give T two (consecutive) edges of curvature $2k$ (and therefore also two edges of curvature $-2k$), as well as straight edges and the edges of curvature $\pm k$, leading to a contradiction, since it has only six sides. We have thus seen that the six edges of T have curvatures $\pm 2k$, $\pm k$, $\pm k$; as before, $\pm 2k$ must be between the two edges of curvature $\pm k$. Thus, T necessarily has the special form $\{-k, -2k, -k, k, 2k, k\}$ of the standard tile.

We now turn to the tilings implied by the shape of this prototile T. If the two sides of curvature k have the same length, then T has mirror symmetry, and both ways of matching edges lead to the same standard tiling. If the two sides of curvature k have different lengths, then each has the same length as exactly one of the sides of curvature $-k$, and there is a unique possible tiling by copies of T. In one case, this is the standard tiling by translations, which we have already ruled out. In the other case, whenever we have a pair of tiles matched across edges of curvature $\pm k$, one is flipped relative to the other. However, we can still find a strip of tiles, as in figure 5, matched across edges of curvature $\pm k$, where the curvatures above the strip are all positive and those below it are all negative. Within the strip, successive tiles are flipped relative to each other, but each tile is translated to the tile two positions further along the strip. This means that the same argument used above applies again, ruling out this tiling when $k>0$.

(iii) We may now assume that every vertex v has at least one straight edge, and thus has curvatures $\langle 0, k, -k\rangle$ (or $\langle 0, -k, k\rangle$) for some $k \geqslant 0$.

Suppose some vertex has $k > 0$. Then the prototile T has two consecutive edges of curvature k (or $-k$), as well as straight edges. Thus, its six curvatures must be 0, $\pm k$ (twice each) and it follows that T is a standard tile $\{0, k, k, 0, -k, -k\}$. Depending on the lengths of the two edges of curvature k, T may or may not have mirror symmetry, and may tile in the standard way or only with flips. But in any case, the matching across the two straight edges must be by translation (without a flip) to avoid an illegal vertex. As in figure 5, this is inconsistent with having all positive curvatures k to one side and all negative curvatures $-k$ to the other. (This tiling with flips is also isohedral, so again we could instead apply lemma 4.1.)

We conclude that $k = 0$. Every vertex has three straight edges. All sides of the prototile T are straight, and T is an equiangular hexagon with opposite sides equal in length. It tiles the plane in a unique way, doubly periodic by translations.

■

4. Open questions

In the above, we have only dealt with foams in equilibrium without external forces. Although theorem 2.1 will not hold if such forces are present, we have pointed out that the notion of a (uniform) pressure gradient remains useful. In relation to theorem 3.1, it would be interesting to explore the following question.

> What are the possible congruent foams in $\mathbb{R}^2$ when external forces are admitted, allowing the condition of 120° angles to be violated?

Even in the absence of external forces, theorem 2.1 leaves open further questions concerning structures which have only partial periodicity, such as the following.

> Does a singly periodic equilibrium foam in $\mathbb{R}^2$ necessarily have periodic pressures?
>
> Does a singly or doubly periodic foam in $\mathbb{R}^3$ necessarily have periodic pressures?

The most challenging open questions concern analogues of theorem 3.1 beyond the planar case. The theorem as such does *not* generalize to three (or higher) dimensions.

We can understand theorem 3.1 to be the combination of five results: in the plane

(a) any monohedral foam is isohedral,

(b) any isohedral foam has equal-pressure bubbles,

(c) any isohedral foam is a tiling by translation,

(d) any isohedral foam has six-sided bubbles,

(e) any foam tiling by translation is a (possibly sheared) hexagonal honeycomb.

In three dimensions, we will see that the analogues of (c) and (d) are false, and we conjecture that the analogues of (a) and (e) are true. We will first show that (b) is true in any dimension.

Lemma 4.1. *An isohedral foam, in Euclidean space of any dimension, has equal-pressure bubbles.*

Proof. The tiles in any isohedral tiling are related by some crystallographic group G of symmetries (Grünbaum & Shephard 1987); this implies that the tiling is fully periodic, with a finite number of tiles in a fundamental cell for the translational lattice. For an isohedral foam, theorem 2.1 says that there is therefore no change in pressure from any bubble to a translated copy. Suppose now there were to be a difference δ in pressure between two bubbles B_0 and B_1. Because the foam is isohedral, these bubbles are related by some congruence α, so that $B_1 = \alpha(B_0)$. Since α is a congruence of the whole foam, we must see the same pressure difference δ between successive copies B_0, $\alpha(B_0)$, $\alpha^2(B_0)$, etc., of this bubble. But it is known (Auslander 1965) that repeating α some number of times must give a lattice translation (because the crystallographic group G has a translation group as a subgroup of finite index). Hence, there is some finite k such that $\alpha^k(B_0)$ is a translation of B_0 (or perhaps B_0 itself). The pressure difference between B_0 and $\alpha^k(B_0)$ is $k\delta$, but is also zero by theorem 2.1. Thus, in fact $\delta = 0$. ■

We could generalize remark 2.2 as follows. Whenever a foam structure (in any ambient space) has a symmetry group G, there is then a map $\phi\colon G \to \mathbb{R}$ such that any two bubbles related by a symmetry $g \in G$ differ in pressure by $\phi(g)$. Our lemma above deals with the case when the ambient space is a flat torus; its proof uses theorem 2.1 to deduce that $\phi \equiv 0$.

In three dimensions, the Kelvin foam and the Williams foam (Williams 1968) are examples of isohedral (and hence equal-pressure) foams with 14-sided bubbles. However, the Williams bubbles do not tile by translation alone, but by screw motions instead. This foam is therefore a counter-example to (c), an isohedral foam which is *not* a tiling by translation.

Kusner (1992) studied equal-pressure foams in $\mathbb{R}^3$, showing that the average number of faces per bubble is at least 13.397. Thus, in a monohedral equal-pressure foam (where this average is an integer: the number of faces on the prototile), each bubble has at least 14 faces. By lemma 4.1, this conclusion applies to any isohedral foam.

However, the bubbles in an isohedral foam in three dimensions need not necessarily have 14 sides. In fact, it is known that there are isohedral tilings of $\mathbb{R}^3$ by convex polyhedra (called 'stereohedra' or 'plesiohedra') with up to 38 faces. Some of these examples with up to 18 faces, including the β-Sn dual structure (O'Keeffe & Sullivan 1998), can be numerically relaxed into equilibrium isohedral foam structures. Presumably this is not true for all such tilings, but the examples do show that there is no good analogue of statement (d) in three dimensions. Since this property had considerably simplified our argument in the planar case, this poses a major obstacle to extending anything like theorem 3.1.

Although, as we have seen, (c) and (d) fail in three dimensions, we do conjecture that analogues of (a) and (e) should be true.

Conjecture 4.2. *A congruent (monohedral) foam in $\mathbb{R}^3$ is necessarily isohedral (and therefore has equal-pressure bubbles).*

Conjecture 4.3. *Any isohedral foam tiling by translation in* $\mathbb{R}^3$ *is necessarily a Kelvin foam (possibly sheared).*

Note that some other characterizations of the Kelvin foam, and related conjectures, appear in Sullivan & Morgan (1996) and Kusner & Sullivan (1996).

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