OPEN PROBLEMS IN SOAP BUBBLE GEOMETRY

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The Burlington Mathfest in August 1995 included an AMS Special Session on *Soap Bubble Geometry*, organized by Frank Morgan. At the end of the session, participants were asked to pose open problems related to bubble geometry. We have collected those problems here, adding a few introductory comments. Participants in the special session included the following:

Fred Almgren, Princeton U.
Megan Barber, Williams C.
Ken Brakke, Susquehanna U.
John Cahn, NIST
Joel Foisy, Duke U.
Christopher French, U. Chicago
Scott Greenleaf, SUNY Stony Brook
Karsten Große-Brauckmann, Bonn
Joel Hass, UC Davis
Aladár Heppes, Budapest
Michael Hutchings, Harvard U.
Jenny Kelley, Rutgers U.
Andy Kraynik, Sandia
Rob Kusner, U. Massachusetts
Rafael Lopez, Granada
Joe Masters, U. Texas
Helen Moore, Bowdoin C.
Frank Morgan, Williams C.
Ivars Peterson, Science News
Robert Phelan, Dublin
Joel Shore, McGill U.
John Sullivan, U. Minnesota
Italo Tamanini, Trento
Jean Taylor, Rutgers U.
Jennifer Tice, Williams C.
Brian Wecht, Williams C.
Henry Wente, U. Toledo
Brian White, Stanford U.

1. Bubble Clusters

The bubble cluster problem asks for the least-area way to enclose and separate k given volumes in an ambient space like $\mathbb{R}^n$. Minimizers are known to exist in the plane [Mor3] and in higher dimensions [Alm2], but little is known about their structure; see [Mor5, Ch. 13]. We can also consider related problems where some other surface energy is minimized.

Problem 1 (Almgren)**.** Is there any stable cluster of bubbles in $\mathbb{R}^3$ with some bubble being topologically a torus?

Conj (Cahn)**:** This could be studied experimentally with sapphire crystals.

Note (Sullivan)**:** Sculpting the complement of a solid torus out of shaving cream suggests that the answer must be yes.

Problem 2 (Sullivan)**.** We construct the standard clusters of k bubbles in $\mathbb{R}^n$ ($k \leq n+1$) as follows. Start with a regular k-simplex in S^{k-1}, and lift this to S^n along longitudes. Then consider different stereographic projections of this complex into Euclidean space; they have S^{n-k} symmetry.

Conj: For any k prescribed volumes, there is a unique standard k-bubble with those volumes.

Conj: This standard k-bubble is uniquely area-minimizing.

Conj: For $n = 3$, these are the only strictly stable k-bubbles ($k \leq 4$). Here we allow bubbles with disconnected regions; strictly stable means that the second variation is positive definite.

Conj: There is a strictly stable cluster of six bubbles with a non-spherical interface (shown on the front cover of [Mor5]), but no such cluster with fewer bubbles.

Note: Since the session, Angel Montesinos [MA] has proved the first conjecture.

Problem 3 (Kusner)**.** Many bubble clusters in $\mathbb{R}^3$, like the standard ones of up to four regions, are made of spherical pieces meeting according to the Plateau rules.

Q: Is any such cluster necessarily stable in the sense of having non-negative second variation?

Note (Morgan)**:** More general equilibrium clusters are locally stable [Mor7, §2]; this probably could be computed directly as in [Cha, §1.3].

Problem 4 (Heppes)**.** Let $A_X(v_1, \dots, v_k)$ denote the area of a minimizing k-bubble of volumes v_i in a Riemannian manifold X.

Q: Is $A_{\mathbb{R}^n}$ strictly increasing in each volume?

Q: Is $A_{\mathbb{R}^n}$ strictly concave?

Note (Hutchings)**:** If $k = 2$ and X is $\mathbb{R}^n$ or S^n or hyperbolic space H^n, then A_X is known to be strictly concave [Hut]. For $\mathbb{R}^n$ and H^n it is strictly increasing for $k = 2$. Very little is known about either question for $k > 2$.

Note (Morgan)**:** An affirmative answer to the first question would imply that the pressure inside each region is nonnegative.

Problem 5 (Heppes)**.** In any optimal partition of a convex domain into two regions, both are connected. However, the best partition of a non-convex domain into two need not use connected regions.

Conj: The regions in a minimizing cluster in $\mathbb{R}^n$ are always connected.

Note (Hutchings)**:** Some results are known for $k = 2$ [FAB$^+$, Hut, Mas].

Problem 6 (Greenleaf, Barber, Tice, Wecht)**.** Suppose we consider clusters made of immiscible fluids with different surface tensions, so that the cost of each interface is a (different) multiple of its area.

Q: What is the shape of such a cluster?

Note: This is open even for the case of planar double bubbles.

Problem 7 (Wecht, Shore, Barber, Tice)**.** A double crystal in $\mathbb{R}^3$ is a double bubble minimizing not area but a surface energy depending on orientation, while enclosing and separating two volumes.

Q: What do double crystals look like, in the case of a cubic Wulff shape, and in the general case?

Conj: The solution uses only those surface orientations which appear on the Wulff shape.

Q (Sullivan)**:** What if we use different cost functions for the three different surfaces in the double crystal?

Note (Morgan)**:** The existence of a minimizing cluster of k regions is known for arbitrary norms in arbitrary dimensions. In the plane, such a cluster consists of norm-minimizing curves (presumably straight lines) and portions of the Wulff shape [MFG].

Problem 8 (Taylor, Almgren)**.** Is a crystal on a table with gravity necessarily convex? (See [Tay] and [Bro, Prob. 8.4].)

2. Cones and Singularities

If a surface minimizes area in some sense then so should its tangent cone at any point, so understanding minimizing cones is essential to describe singularities. For area-minimizing hypersurfaces, there are no singularities below dimension eight, but the cone over $S^3 \times S^3$, for instace, is minimizing [BDG]. But for classes of surfaces including triple junctions, such as soap films, many more cones are possible, even in low dimensions.

Problem 9 (Brakke)**.** We call a minimizing (soap-film) cone polyhedral if it is made entirely of flat pieces. The Plateau singularities, for instance, are polyhedral. Brakke has classified them in dimension four [Bra2]; Sullivan has listed the candidates in higher dimensions [Sul2].

Q: What are the minimizing polyhedral cones in dimensions five through seven?
Q: What are the minimizing polyhedral cones in all higher dimensions (where certainly also non-polyhedral cones exist)?

Problem 10 (Sullivan)**.** There are of course non-polyhedral minimizing cones in higher dimensions, without triple junctions.

Q: Are there any nonpolyhedral minimizing soap film cones in dimensions four through seven?
Note: White has ruled out many nonpolyhedral minimal cones in $\mathbb{R}^4$, including Hopf lifts of the Plateau cones, by an argument [Whi] which shows the link in S^3 of a minimizing cone is either polyhedral, or includes both a disk surface and a nongeodesic triple curve.
Q: What are some examples in higher dimensions which use triple junctions, and thus are not cones on smooth manifolds?

Problem 11 (Tamanini)**.** Let $k(n)$ be the maximum number of regions that can meet at one point in an area-minimizing partition in n dimensions. Tamanini has shown [MT, TC] that $k(n) < \infty$. Brakke's calibration of hypercube cones [Bra1] shows $k(n) \geq 2n$.

Q: What is the growth of $k(n)$? Is it strictly increasing?
Conj (Morgan)**:** The function $k(n)$ grows exponentially.
Q (Kusner)**:** What about partitions minimizing other surface energies? (Note that the growth is exponential for the Manhattan metric.)

Problem 12 (Morgan)**.** Seven possibilities for boundary singularities of soap films in $\mathbb{R}^3$ are listed in [Mor2]; a more detailed classification into ten cases is given in [LM2, §5.2].

Q: Is this a complete list of boundary singularities?
Q (Kusner)**:** What if the boundary itself is singular; what happens as a family of embedded boundaries becomes immersed?
Q (Brakke)**:** What if we also allow soap films which touch only part of their boundary wires? There are at least two new cases then.

Problem 13 (White)**.** What is the smallest density (greater than one) of a minimal cone in $\mathbb{R}^n$. (We could add conditions like area-minimizing or isolated singularity if desired; the problem could be posed for stationary integral varifolds.)

Note: For $n = 2$ the answer is $3/2$, as in the cone Y. This density is known to exist (and be bounded away from one) for each dimension [All, §8], and is conjectured to approach $\sqrt{2}$ as $n \to \infty$, as in the example of the cone over $S^m \times S^m$.
Note: This least density cone has the form $\mathbb{R}^p \times C$, where C is regular away from the origin.
Note (Morgan)**:** For $n = 6$, there are two minimal cones with density $3/2$, presumably the smallest nontrivial density: the cone on $S^2 \times S^2$, and $Y \times \mathbb{R}^4$.

Problem 14 (Kusner)**.** What singularities are allowed in soap films in three-dimensional orbifolds, or more generally in cone manifolds?

Note: Kusner and Sullivan have completed the classification for certain kinds of orbifold points [KS].

3. Partitions and Foams

Kelvin's problem asks for the optimal partition of space into equal-volume cells. The solution is expected to have the geometry of a foam, but not even existence is known for such infinite clusters. (See [Bro, Prob. 1.2].)

Problem 15 (Morgan)**.** Do least-area partitions of $\mathbb{R}^n$ into unit volumes exist? What is the right definition? What regularity holds?

Problem 16 (for the bees)**.** Are regular hexagons the optimal way to partition the plane into equal areas?

Note (Sullivan)**:** Results of Fejes Tóth [FT1, §III.9] [FT2, §26] together with a truncation argument [Mor6] imply this is best if the cells have equal pressure.

Note (Heppes)**:** Under the assumption of equal pressure, some optimal partitions are also known for the sphere and hyperbolic plane.

Problem 17 (Phelan)**.** Weaire and Phelan [WP] have described a very efficient foam, based on the A15 structure.

Q: Is this the optimal partition of space into equal volumes?

Note (Sullivan)**:** It is provably better than Kelvin's foam [AKS].

Q (Sullivan)**:** Can we prove the existence of a foam in the A15 pattern?

Problem 18 (Sullivan)**.** Kelvin's foam, based on the BCC lattice, is not the optimal partition of space into equal volumes, though it is the best with its combinatorics and symmetry [AKS]. It might be optimal for various other restricted problems, like that considered by Choe [Cho1] who proved that any three-manifold has a least-area fundamental domain. (See also [KS].) The following conjectures are successively stronger.

Conj: The Kelvin cell is the Choe cell for the BCC torus.

Conj: It is the least area fundamental domain for any flat torus of unit volume.

Conj: The Kelvin foam is the best partition with congruent cells.

Conj: It is the best partition with equal-pressure cells.

Problem 19 (Heppes)**.** Suppose we have an optimal cluster or partition of a region in the plane, and extend it to a partition of a slab in space by crossing with a short interval $[0, \epsilon]$.

Q: Is this an optimal partition of the slab?

Q: Suppose we wish to divide a very long (vertical) cylinder or prism into two equal-volume halves. Is the optimal way a horizontal slice halfway up?

Problem 20 (Kraynik)**.** The regular hexagonal foam in the plane has equal shear modulus in all directions. Weaire, Fu and Kermode [WFK] have conjectured that no two-dimensional froth (with average bubble area one) has greater shear modulus in any direction.

Problem 21 (Sullivan)**.** A cell in an equal-pressure foam in $\mathbb{R}^3$ is bounded by minimal surfaces meeting according to Plateau's rules.

Conj: There are only finitely many possible combinatorial types for such cells. In particular, it seems unlikely that tetrahedra can occur. Probably not even a dodecahedron can occur, which would imply that none of the foams based on TCP structures can be made equal-pressure.

Note: Several types of cells are known, including Kelvin's truncated octahedron, and all have at least 14 faces. Kusner has shown [Kus] that the cells in an equal-pressure foam must have at least 13.39 sides on average.

Note: A dihedral cell could exist only if there is a boundary curve in $\mathbb{R}^3$ which bounds two minimal surfaces meeting at 120° angles along its entire length.

4. Isoperimetric Inequalities

The sphere is the least-area way to enclose a given volume. More general isoperimetric inequalities deal with surfaces of higher codimension spanning a given boundary, and thus can only apply to surfaces which are in some sense minimal or of constant mean curvature. When a given boundary is not a manifold, there are different notions in which a surface might span it.

Problem 22 (Almgren)**.** Theorems of Allard [All] and Almgren [Alm1] give a constant C such that if B is a bubble cluster with area A and boundary of length L then

$$A \leq C \left(\int_B H + L \right)^2 .$$

Q: What is the best constant C? This is an open question even if B is a smooth surface of mean curvature $H = 0$, although the expected value $C = 1/4\pi$ has been proved in many special cases [Oss, p. 147]; see also [Cho2, CG, Cho3].
Q: What happens for other norms? (There are constants C for norms sufficiently close to area [Alm3]; what about for norms farther away?)
Q: What about higher dimensions? (The theorems of Allard and Almgren above are general in dimension and codimension.)

Problem 23 (Hutchings)**.** Gromov ([Gro]; see also [Ber, §12.11.4] or [BM]) gave an interesting proof of the isoperimetric inequality, based on an idea of Knothe.

Q: Can this be extended to handle the case of more than one region?
Q: Can the vector field used be chosen canonically?
Note: In the two-dimensional case (for one region), the vector field can be chosen as the gravitational field of a mass filling the region with uniform density.

Problem 24 (Morgan)**.** Melzak [Mel] has conjectured that the unit volume polyhedron with shortest total edge length is an equilateral triangular prism.

Note: Morgan [Mor4, §1.4] has offered \$200 for a solution. The existence of a minimizer is not known. Some recent evolver experiments [Kel] by Jenny Kelley at Rutgers and Craig Carter at NIST seem to support the conjecture.

Problem 25 (Morgan)**.** Morgan and Lawlor have shown [LM1] that the cone on the regular tetrahedron is the least area way to separate the four regions.

Q: Is it the smallest soap film (meaning an $(M, 0, \delta)$-minimizing set) having that entire frame as boundary? (See [Mor1] and [LM1, fig. 1.1.1].)
Q: On a frame consisting of two rectangles sharing an edge, with small external dihedral angle (like an open book), is the obvious soap film the smallest one?

Problem 26 (Brakke)**.** For a regular octahedral wire frame, what is the least-area soap film separating the eight regions? What if we allow fractional density soap films? (See [Bra3].)

5. CMC Surfaces, and Curvature Bounds

The remaining problems deal with surfaces of constant mean curvature, and with bounds on curvature for area-minimizing surfaces.

Problem 27 (Lopez)**.** If Γ is a planar convex curve with length less than 2π, is there a graph with boundary Γ and constant mean curvature one?

Note: When Γ is a round circle, the answer is yes. If Γ has length less than $\sqrt{3}\pi$, the answer is yes [LoM1]. See [Ser] for general results about existence of CMC graphs.

Note: Given a planar convex curve Γ, there is a value $V(\Gamma)$, such that any CMC surface with boundary Γ and algebraic volume less than $V(\Gamma)$ must be a graph [LoM1].

Q: What is $V(\Gamma)$?

Note: If the surface is a disk and Γ is a unit circle, the best value $V(\Gamma)$ is $2\pi/3$ [LoM1].

Q: Is this still true if the surface is not a disk?

Problem 28 (Lopez)**.** Is an embedded CMC surface (or an immersed CMC disk) with boundary a round circle necessarily a spherical cap? (See [Bro, Prob. 1.4].)

Note: Partial results are known in the embedded case with convex planar boundary [BEMR], using techniques from [KKS] and [Wen]. The answer is yes in the immersed CMC disk case [LoM2] if the area of the surface is less than that of the small spherical cap with the same mean curvature.

Note (Kusner)**:** There are known immersed examples of higher topological type [Kap].

Problem 29 (Moore)**.** Consider a minimal submanifold $M^k \subset \mathbb{R}^n$. If it has finite total scalar curvature, then its tangent cone at infinity is a sum of flat planes through the origin [And]. If it is area-minimizing, its tangent cone must also be area-minimizing.

Note: Any complex submanifold $M \subset \mathbb{C}^N$ is area-minimizing; the total scalar curvature of M is finite if its complex dimension is one, but infinite if the dimension is greater [Moo].

Note: The special Lagrangian submanifolds $M^k \subset \mathbb{R}^{2k}$ constructed by Lawlor [Law] are area-minimizing and have finite total scalar curvature.

Note: There are no known examples of (complete) nonplanar area-minimizing hypersurfaces with finite total scalar curvature.

Q: Is an area-minimizing surface of finite total scalar curvature necessarily planar for $k > n/2$?

Note (Morgan)**:** For $k > n/2$, if there are two distinct planes in the tangent cone at infinity, they meet along a line. If $k = n - 1$, a tangent cone with two distinct planes is not minimizing.

Problem 30 (Sullivan)**.** Curvature bounds for area-minimizing hypersurfaces in low dimensions are given in [SSY]; these say that principal curvatures are bounded by C/r at any point a distance r from the boundary.

Q: What is the best constant C?

Note: In [Sul1] some upper bounds are computed explicitly, but for surfaces in $\mathbb{R}^3$ we find $C \approx 800000$, while the best known examples have $C \approx 2$.

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