Criticality for the Gehring Link Problem

Jason Cantarella,1, * Joseph H.G. Fu,1, † Rob Kusner,2, ‡ John M. Sullivan,3,4, § and Nancy C. Wrinkle1,5, ¶

1Department of Mathematics, University of Georgia, Athens, GA 30602
2Department of Mathematics, University of Massachusetts, Amherst, MA 01003
3Department of Mathematics, University of Illinois, Urbana, IL 61801
4Institut für Mathematik, Technische Universität Berlin, D–10623 Berlin
5Department of Mathematics, Northeastern Illinois University, Chicago, IL 60625

(Dated: November 19, 2003; Revised: February 12, 2004)

In 1974, Gehring posed the problem of minimizing the length of two linked curves separated by unit distance. This constraint can be viewed as a measure of thickness for links, and the ratio of length over thickness, as the (Gehring) ropelength. In this paper we refine Gehring’s problem to deal with links in a fixed link-homotopy class: we prove ropelength minimizers exist and introduce a theory of ropelength criticality.

Our balance criterion is a set of necessary and sufficient conditions for criticality, based on a strengthened, infinite-dimensional version (Theorem 5.4) of the Kuhn–Tucker theorem. We use this to prove that every critical link is $C^1$ with finite total curvature. The balance criterion also allows us to explicitly describe critical configurations (and presumed minimizers) for many links including the Borromean rings. We also exhibit a surprising critical configuration for two clasped ropes: near their tips the curvature is unbounded and a small gap appears between the two components. These examples reveal the depth and richness hidden in Gehring’s problem and our natural extension of it.

Keywords: Gehring link problem, link homotopy, link group, ropelength, ideal knot, tight knot, constrained minimization, Mangasarian–Fromovitz constraint qualification, Kuhn–Tucker theorem, simple clasp, Clarke gradient

1. INTRODUCTION

Suppose that $A$ and $B$ are disjoint linked Jordan curves in $\mathbb{R}^3$ which lie at a distance 1 from each other. Show that the length of $A$ is at least $2\pi$.

—Fred Gehring, 1974

Gehring’s problem, posed in a conference proceedings [CH74], was soon solved by Marvin Ortel. Because Ortel’s elegant solution was never published, we reproduce it here with his permission: Fix any point $a \in A$; the cone on $A$ from $a$ is a disk spanning $A$. Since $A$ and $B$ are linked, $B$ meets this disk at some point $b \in B$, lying on a chord of $A$. Because Dist$(A, b) \geq 1$, projecting $A$ to the unit sphere around $b$ does not increase its length. The projection is a closed curve joining two antipodal points on the sphere, and so has length at least $2\pi$. (Alternate proofs, and generalizations to linked spheres in higher dimensions, were published in [ES76, Oss76, Gag80, Gag81].)

The unique minimizing configuration for Gehring’s problem is a Hopf link consisting of two congruent circles in perpendiculair planes, each passing through the other’s center. This leads to a natural question: what are the length-minimizing shapes of other link types when the different components stay unit distance apart? This constraint prevents different components from crossing each other, but we cannot expect to fix the link type exactly. Instead, the natural setting for this problem is Milnor’s notion of link homotopy: two links are link-homotopic if one can be deformed into the other while keeping different components disjoint. Clearly one link can be deformed into another while keeping all components at unit distance if and only if they are link homotopic.

We will define the Gehring thickness of a link to be the minimum distance between different components. The problem we consider is then to minimize length in a link-homotopy class, subject to the constraint of fixed Gehring thickness. Equivalently, we could minimize the Gehring ropelength of the link, meaning the quotient of length over thickness.

In [CKS02], we found length-minimizing links under a similar constraint: that a normal tube of diameter one around the link stay embedded. It is easy to see that the examples constructed there are also global minima (in their respective link-homotopy classes) for the Gehring problem. The focus of this paper will be on critical configurations. Our main result is a balance criterion (Theorem 6.1, Corollary 6.3), which states that a link is Gehring-ropelength critical if and only if the tension force in the curve is balanced by a system of compressive forces between pairs of points on different components of $L$ realizing the minimum distance.

This balance criterion is based on an improved, infinite-dimensional version (Theorem 5.4) of the Kuhn–Tucker theorem on constrained optimization, which is essentially a very general method of Lagrange multipliers. The other key technical element is a careful application of Clarke’s differentiation theorem for min-functions (Theorem 3.1).

The direct method shows that there is a (rectifiable) minimizer for Gehring-ropelength in each link-homotopy class. An interesting problem is to determine the regularity of these minimizers or other critical points. The previously known minimizers were $C^{1,1}$ but not $C^2$. Our balance criterion allows us to prove that all Gehring-ropelength-critical curves are $C^1$ with finite total curvature (Proposition 6.5).

We then consider generalized links which may include open...
components with constrained endpoints, or which avoid fixed obstacles. After extending our balance criterion and existence results to this setting, we analyze the problem of the simple clasp. A clasp consists of two linked arcs whose endpoints are constrained to parallel planes (as in Figure 10). A generalization to clasp of different opening angles provides a model for the strands of rope in a woven cloth or net. The balance criterion lets us construct explicit critical configurations (Theorem 9.5) of these generalized links; we conjecture they are the length-minimizers subject to the constraint that the arcs remain at unit distance from each other. Our critical clasp has a number of surprising features, including a point of infinite curvature and a small gap (at the center of the clasp) between the tubes around the two components. This configuration is $C^{3,2/3}$, and may represent the worst regularity of any Gehring-critical curve.

We end by constructing a ropelength-critical configuration (and presumed minimizer) for the Borromean rings. In all the other known critical configurations for closed links, each component is a convex plane curve built from straight segments and arcs of unit circles. In our Borromean rings, the components are still planar, but are nonconvex, and are built from different pieces including parts of a clasp curve. In a sense, this is the first nontrivial example known of a ropelength-critical link.

Our methods will have a number of other applications. In particular, we have used them to describe critical configurations for the “standard” ropelength problem for knots and links: minimize the length of a $C^1$ link subject to the constraint that the normal neighborhood of unit diameter remains embedded. We will publish these results in a sequel [CFK+04] to the current paper. We can also consider minimization not of length but of other objective functions like elastic bending energy, again subject to a thickness constraint. Analogs of our balance criterion may be useful in describing other flexible mechanisms, such as thick surfaces.

We note that von der Mosel and Schuricht [vdMS03] have used a similar approach (via Clarke’s theorem and a functional-analytic version of Lagrange multipliers) to derive necessary, but not sufficient, conditions for criticality for the ropelength functional of [CKS02]. We treat the same functional in the (forthcoming) [CFK+04], and will offer a comparison of the two methods there. We also note that Starostin has announced [Sta03] an independent derivation of the tight clasp of Section 9, though he does not prove that it is Gehring-critical.

2. GEHRING THICKNESS FOR CLOSED LINKS

In order to reformulate Gehring’s problem, we first establish some basic terminology. Remember that a compact, oriented 1-manifold-with-boundary $M^1$ is a finite union of components, each of which is homeomorphic to a circle $S^1$ or an interval $[0, 1]$.

**Definition.** A parametrized curve is a map from a compact, oriented 1-manifold-with-boundary $M^1$ to $\mathbb{R}^3$. Two parametrized curves are equivalent if they differ by an orientation-preserving reparametrization (i.e., by composition with an orientation-preserving self-homeomorphism of $M^1$). A curve $L$ in $\mathbb{R}^3$ is an equivalence class of parametrized curves. We say $L$ is closed when each component of its domain $M^1$ is a circle, that is, when its boundary $\partial L$ is empty.

Even though our curves may have self-intersections, we will usually refer to points on the curve as if they were simply points of its image in $\mathbb{R}^3$. The meaning should be clear from context.

The length $\text{Len}(L)$ of any curve $L$ is defined to be the supremal length of all polygons inscribed in $L$. A curve has finite length, or is rectifiable, if and only if it has a Lipschitz (i.e., $C^{0,1}$) parametrization. One such parametrization is then by arclength $s$. Any rectifiable curve has a well-defined unit tangent vector $T = dL/\|ds\|$ almost everywhere.

**Definition.** The Gehring thickness $\text{GThi}(L)$ of a curve $L$ is the minimal distance between points on different components of $L$. This is the supremal $\varepsilon$ for which the $(\varepsilon/2)$-neighborhoods of the components of $L$ are disjoint.

For now, we will consider only the case of closed curves (where each component is a circle); we will deal with generalized links (with endpoint constraints) later in Section 8. So suppose we start with a closed curve $L$ and we want to minimize length under the constraint that the Gehring thickness remains at least one. (Since we can rescale any link to have $\text{GThi} \geq 1$, this problem is the same as minimizing ropelength, the quotient of length by Gehring thickness.) The thickness constraint naturally prevents different components from passing through each other, but does not prevent any given component from changing its knot type through self-intersections. This is the setting for Milnor’s work on link homotopy:

**Definition.** A link is a closed curve with disjoint components. The link-homotopy class of a link $L$, denoted $[\lbrack L \rbrack]$, is the set of curves homotopic to $L$ through configurations that keep different components of $L$ disjoint.

Note that, for our purposes, configurations of $L$ where some components have self-intersections are still considered links, and are included in $[\lbrack L \rbrack]$.

For two-component links, Milnor [Mil54] showed that linking number is the only link-homotopy invariant. For links of many components, the topological situation is more complicated, but a complete classification of links up to link homotopy was provided by Habegger and Lin [HL90]. We will prove in Section 6 that in every link-homotopy class there is a minimizing curve. We show these minimizers are always $C^1$, though some of our examples suggest that they may not always have bounded curvature.

3. THE DERIVATIVE OF GEHRING THICKNESS

We want to define critical configurations of $L$ subject to the thickness constraint $\text{GThi}(L) \geq 1$. Because $\text{GThi}$ is defined as the minimum of a collection of distances between points
on different components, the equation $\text{GThi} \geq 1$ acts like a collection of many constraints. To make this notion precise, we will apply a theorem of Clarke to compute the derivative of $\text{GThi}$ as we vary the curve $L$.

Given any curve $L$, let $L(2)$ be the compact set of all unordered pairs of points on distinct components of $L$. The Gehring thickness of $L$ is simply the minimum over $L(2)$ of the distance function $\text{Dist}(x, y) := |y - x|$. We often want to consider a continuous deformation $L_t$ of a curve $L$: for any parametrization $f$ of $L$, a continuous family $f_t$ of parametrized curves with $f_0 = f$. (When we reparametrize $L_t$, we apply the same reparametrization to $L_t$ at all times $t$.) We assume that $L_t$ is $C^1$ in $t$; the initial velocity of $L_t$ will then be given by some (continuous, $\mathbb{R}^3$-valued) vectorfield $\xi$ along $L$. We let $\text{VF}(L)$ denote the space of all such vectorfields. Formally, these are sections of the bundle $f^*\mathbb{T}\mathbb{R}^3$ pulled back from the tangent bundle of $\mathbb{R}^3$ by the parametrization $f$ of $L$. Identifying any tangent space to $\mathbb{R}^3$ with $\mathbb{R}^3$ itself, this is simply a map from the domain $M^1$ to $\mathbb{R}^3$. Again, when we reparametrize a curve $L$, we apply the same reparametrization to any vectorfield $\xi$.

Consider a curve $L$ with $\text{GThi}(L) > 0$. If $L_t$ is a continuous deformation of $L$, with initial velocity given by some $\xi \in \text{VF}(L)$, then for each pair $\{x, y\} \in L(2)$, we clearly have

$$\delta_\xi \text{Dist}(x, y) := \frac{d}{dt} |y - x|_{|t=0} = \frac{1}{|y - x|} \langle \xi_y - \xi_x, y - x \rangle.$$ (Even if $L$ is not embedded, the condition $\text{GThi}(L) > 0$ implies $x$ and $y$ cannot coincide in $\mathbb{R}^3$, so this formula is meaningful.)

The Clarke gradient. A function like $\text{GThi}$, defined as the minimum of a compact family of smooth functions, is sometimes called a min-function. Clarke’s differentiation theorem for min-functions says that—just as in the case when the compact family is finite—the derivative of a min-function is the smallest derivative of those smooth functions that achieve the minimum. More precisely, specializing [Cla75, Thm. 2.1] to the case we need, we have:

**Theorem 3.1 (Clarke).** Suppose for some compact space $K$ and some $\varepsilon > 0$, we have a family of $C^1$ functions $f_k : (-\varepsilon, \varepsilon) \to \mathbb{R}$, for $k \in K$. Suppose further that $f_k(t)$ and $f_k'(t)$ are lower semicontinuous on $K \times (-\varepsilon, \varepsilon)$. Let $f(t) := \min_{k \in K} f_k(t)$. Then $f$ has one-sided derivatives, and

$$\frac{df}{dt}\bigg|_{t=0} = \min_{k \in K_0} f_k'(0),$$

where $K_0 := \{k \in K : f_k(0) = f(0)\}$ is the set of $k$ where the minimum in the definition of $f$ is achieved for $t = 0$. $\square$

To apply this theorem to thickness, suppose we have a variation $L_t$ of the curve $L$, and let $\xi \in \text{VF}(L)$ be its initial velocity. The Gehring thickness $\text{GThi}(L)$ is written as a minimum over $K = L(2)$ of the pairwise distance. Clarke’s theorem picks out those pairs achieving the minimum: $K_0$ is the set of pairs achieving the minimum distance $\text{GThi}(L)$.

**Notions from rigidity theory.** In rigidity theory (see [RW81]), the vertices of a tensegrity framework are joined by bars whose length is fixed, cables whose length can shrink but not grow, and struts whose length can grow but not shrink. Thus, we borrow the term “strut” to describe the pairs in $K_0$:

**Definition.** A (unordered) pair of points $x$ and $y$ on different components of $L$ is a Gehring strut if $|y - x| = \text{GThi}(L)$. The set of all Gehring struts of $L$ is denoted $\text{GStrut}(L)$.

Struts correspond to points of contact between tubes around the different components of $L$. Our balance criterion will show how the segment $xy$ can be viewed as carrying a force pushing outwards on its endpoints.

Applying Clarke’s theorem to Gehring thickness, we get:

**Corollary 3.2.** For any curve $L$, and any variation vectorfield $\xi \in \text{VF}(L)$, the first variation of Gehring thickness is

$$\delta_\xi^+ \text{GThi}(L) = \min_{\text{GStrut}(L)} \delta_\xi \text{Dist}(x, y).$$

$\square$

Note that $\delta_\xi \text{Dist}(x, y)$ is a continuous function of $x$ and $y$, and for any fixed $\{x, y\}$ is a linear function of the variation $\xi$, being the derivative of a smooth function.

So we can collect these into a linear operator $A_G = \delta \text{Dist}$ from $\text{VF}(L)$ to $C(\text{GStrut}(L))$, defined by

$$(A_G \xi)(\{x, y\}) := \frac{1}{|y - x|} \langle \xi_y - \xi_x, y - x \rangle.$$ Borrowing again from finite-dimensional rigidity theory, where the analogous $A$ is called the rigidity matrix, we will call $A_G$ the rigidity operator for Gehring thickness.

The corollary above can be rephrased to say that a variation $\xi$ decreases $\text{GThi}(L)$ to first order if and only if $A_G \xi$ fails to be a nonnegative function on $\text{GStrut}(L)$.

Note that, while the corollary says that Gehring thickness has a directional derivative $\delta_\xi^+ \text{GThi}$ in each direction $\xi$, the operator $\delta_\xi^+ \text{GThi}$ is not linear in $\xi$. For instance, when one component of a Gehring link is between two others, it is easy to have both $\delta_\xi^- \text{GThi} < 0$ and $\delta_\xi^+ \text{GThi} < 0$. We write the superscript $\dagger$ to emphasize that these are only one-sided derivatives. There is, however, a form of superlinearity:

**Corollary 3.3.** For any curve $L$ and any $\xi, \eta \in \text{VF}(L)$, we have

$$\delta_{\xi + \eta}^+ \text{GThi}(L) \geq \delta_\xi^+ \text{GThi}(L) + \delta_\eta^+ \text{GThi}(L).$$

$\square$

**Proof.** This follows immediately from the linearity of $A_G$ and the general fact that $\min(f + g) \geq \min f + \min g$: we have

$$\min A_G(\xi + \eta)(\{x, y\}) = \min \{A_G \xi + A_G \eta\}(\{x, y\})$$

$$\geq \min A_G \xi(\{x, y\}) + \min A_G \eta(\{x, y\}),$$

where the minima are taken over all $\{x, y\} \in \text{GStrut}(L)$. $\square$
We will be interested in the adjoint $A_G^*$ of the rigidity operator, so we will first consider the dual function spaces. By the Riesz representation theorem, we know that $C^*(G\text{Strut})$ is the space of signed Radon measures on the space $G\text{Strut}(L)$ of struts. Similarly $VF^*(L)$ is the space of what we will call forces along $L$, namely vector-valued Radon measures on $L$. The adjoint operator $A_G^*$ associates to any measure on struts a force along $L$. Geometrically, each pair $[x, y]$ acts along the chord $xy$, outwards at each of its endpoints. In formulas,
\[
\int_L \xi \, dA_G^*(\mu) = \int_{G\text{Strut}} A_G\xi \, d\mu = \int_{x \in L} \int_{y \in L} \left( \frac{y - x}{|y - x|} \right) \, d\mu(x, y),
\]
where we have lifted $\mu$ to a symmetric measure $\mu(x, y)$ on ordered pairs.

Physically, we think of $\mu$ as giving the strengths of compressive forces within the struts, and $A_G^*$ as the operation that integrates these strut forces to give a net force along the curve $L$.

4. FIRST VARIATION OF LENGTH, AND FINITE TOTAL CURVATURE

The objective functional we consider in this paper is simply the length $\text{Len}(L)$ of a curve. Since our curves might not be smooth, we need to carefully examine the first variation of length.

Let $L$ be a rectifiable curve with arclength $s$ and tangent vector $T$. Suppose $L_t$ is a variation of $L$ under which the motion of each point $x \in L$ is smooth in time with initial velocity $\xi(x)$, and $\xi \in VF(L)$ is a Lipschitz function of arclength. Then the standard first-variation calculation shows that
\[
\delta \xi \text{Len}(L) = \frac{d}{dt} \text{Len}(L_t) \bigg|_{t=0} = \int_L \langle T, \xi' \rangle \, ds,
\]
where $\xi' = d\xi/ds$ is the arclength derivative, defined almost everywhere along $L$.

If $L$ is smooth enough, we can integrate this by parts to get
\[
\delta \xi \text{Len}(L) = -\int_L \langle T', \xi \rangle \, ds - \sum_{x \in \partial L} \langle \pm T, \xi \rangle.
\]
In the boundary term, the sign is chosen to make $\pm T$ point inward at $x$. In fact, not much smoothness is required: as long as $T$ is a function of bounded variation, we can interpret $T'$ as a measure, and the formula holds in the sense we will now explore.

Following Milnor [Mil50], we recall that the total curvature of a polygon is just the sum of its (exterior) turning angles, and we define the total curvature of any curve to be the supremum of total curvature over all inscribed polygons. A rectifiable curve $L$ has finite total curvature if and only if the unit tangent vector $T = L'(s)$ is a function of bounded variation. Sometimes the space of all such curves is called $W^{1,1}$ or $BV^1$, but we will call it FTC.

If $L \in\text{FTC}$, it follows that at every point of $L$ there are well-defined left and right tangent vectors $T_\pm$; these are equal and opposite except at countably many points, the corners of $L$. (See, for instance, [Roy63, Sect. 5.2].)

If $L \in\text{FTC}$, its tangent $T$ has a distributional derivative $K$ with respect to arclength: a force (an $\mathbb{R}^1$-valued Radon measure) along $L$ that we call the curvature force.

The curvature force has an atom (a point mass or Dirac delta) at each corner $x \in L$, with $K\{x\} = T_+(x) + T_-(x)$. On a $C^2$ arc of $L$, the curvature force is $K = -\int ds$ and is absolutely continuous with respect to the arclength or Hausdorff measure $\int ds = \mathcal{H}^1$.

When $L$ has boundary, we choose to include in $K$ an atom at each endpoint of $L$, with mass 1 and pointing in the inward tangent direction. This means we need no boundary terms in the formula $\delta \xi \text{Len}(L) = -\int_L \langle \xi, dK \rangle$.

We say that a vectorfield $\xi$ along $L$ is smooth if $\xi(s)$ is a smooth function of arclength. (The arclength parameterization of any rectifiable curve is essentially unique, so this makes sense.) The set of all smooth vectorfields will be denoted $VF^\infty(L)$.

The first variation $\delta \text{Len}(L)$ can be viewed as a linear functional on smooth vectorfields $\xi \in VF^\infty(L)$. As such a distribution, it has degree zero, by definition, if $\delta \xi \text{Len}(L) = -\int_L \langle T, \xi' \rangle \, ds$ is bounded by $C \sup_x |\xi|$ for some constant $C$. This happens exactly when we can perform the integration by parts.

We collect these results as:

**Lemma 4.1.** Given any rectifiable curve $L$, the following conditions are equivalent:

(a) $L$ is FTC.

(b) The first variation $\delta \text{Len}(L)$ has distributional degree zero.

(c) There exists a curvature force measure $K$ along $L$ such that $\delta \xi \text{Len}(L) = -\int_L \langle \xi, dK \rangle$.

\[\square\]

An FTC curve $L$ is $C^1$ exactly when it has no corners, that is, when $K$ has no atoms (except at the endpoints). It is furthermore $C^{1,1}$ when $T$ is Lipschitz, or equivalently when $K$ is absolutely continuous (with respect to arclength) and has bounded Radon-Nikodym derivative $dK/\,ds = \kappa N$. In previous work on ropelength (see, for example [CKS02, GMSvdM02]), the thickness measure had an upper bound for curvature built in, meaning that any curve of positive thickness was automatically $C^{1,1}$. This is not true for the Gehring thickness, so we do not expect the same regularity results to hold here.

5. CONSTRAINED CRITICALITY AND THE KUHN–TUCKER THEOREM

We will review constrained minimization problems in a finite setting, before generalizing to the setting we will need for...
Theorem 5.2 (Modified Kuhn–Tucker Theorem). by a Lagrange multiplier theorem (cf. [KT51]): required. However, critical points can be exactly characterized to our definition, an additional regularity hypothesis will be needed. some component of \( g_j \) not canceled by the \( \nabla g_j \), that would give an admissible direction to move which decreases \( f \).

Unlike in the classical Kuhn–Tucker theorem, we do not need additional regularity hypotheses on the point \( p \), which may surprise those familiar with optimization theory. The explanation is that we are interested in critical points, while the classical theorem deals with minima of \( f \). And as we saw above, not every minimum of \( f \) is critical. But just as in the classical theory, criticality will be guaranteed if we add the hypothesis that the Mangasarian–Fromovitz constraint qualification [MF67] holds for a local minimum.

Definition. A point \( p \) is constraint-qualified (in the sense of Mangasarian and Fromovitz) if there is a direction \( v \) such that for all constraints \( g_j \) active at \( p \) we have \( D_v g_j > 0 \).

We note that this condition fails at the point \( p = (1, 0) \) in Example 5.1 above, which was minimal but not critical.

Proposition 5.3. If \( p \) is a local minimum for \( f \) when constrained by \( \{ g_i \geq 0 \} \), and \( p \) is constraint-qualified, then \( p \) is constrained-critical for minimizing \( f \).

We have omitted proofs of the theorem and proposition above, because they are standard and are also special cases of our infinite-dimensional generalizations below.

A generalized Kuhn–Tucker theorem. Note that in Theorem 5.2, the functions \( f \) and \( g_i \) might as well be replaced by linear functions—their differentials at \( p \). We view this as the linear-algebraic core of the Kuhn–Tucker theorem.

We will now derive an infinite-dimensional version, where the linear functional \( f \) is defined on an arbitrary vectorspace \( X \), and the finite family of constraints \( g_i \) is replaced by a family \( A_y \), where \( y \) ranges over some compact space \( Y \).

While our theorem does not mention optimization directly, it will be the engine that drives all of the optimization theorems of this paper.

As usual, we let \( C(Y) \) be the Banach space of continuous functions on \( Y \) with the sup norm \( \| \cdot \| \), and let \( P \subset C(Y) \) be the closed positive orthant consisting of nonnegative functions. Then (by the Riesz representation theorem) their duals are \( C^*(Y) \), the space of signed Radon measures on \( Y \), and \( P^* \subset C^*(Y) \), the positive measures.

Note that any function \( z \in C(Y) \) can be decomposed into positive and negative parts: \( z = z^+ - z^- \) with \( z^\pm \in P \). Then we have \( \|z^-\| = \text{Dist}(z, P) \).

Theorem 5.4. Let \( X \) be any vectorspace and \( Y \) be a compact topological space. For any linear functional \( f \) on \( X \), and any linear map \( A: X \to C(Y) \), the following are equivalent:

(a) There exists \( \varepsilon > 0 \) such that \( \| (A\xi)^- \| \geq \varepsilon \) for all \( \xi \in X \) with \( f(\xi) = -1 \).

(b) There exists a positive Radon measure \( \mu \in P^* \) such that \( f(\xi) = \mu(A\xi) \) for all \( \xi \in X \).

This theorem is comparable to the generalized Kuhn–Tucker theorem of Luenberger [Lue69, p. 249]. His theorem, restated to apply to the linear Gateaux differentials \( (f, A) \) of the original objective and constraint functions on \( X \), says:

Theorem 5.5 (Luenberger). Let \( X \) and \( Z \) be vectorspaces, with a norm given on \( Z \), and let \( P \subset Z \) be a closed convex cone with nonempty interior. Let \( f: X \to \mathbb{R} \) be a linear

![Figure 1: In this illustration of Example 5.1, the admissible region for the two constraints \( g_1 = (x^2 - 1)^3 - z \geq 0 \) and \( g_2 = z \geq 0 \) is shaded. The Mangasarian–Fromovitz constraint qualification fails at the cusp point \( p = (1, 0) \) because \( \nabla g_1 \) and \( \nabla g_2 \) are equal and opposite there.](image-url)
functional and \( A : X \to Z \) be a linear map. Assume that whenever \( A \xi \in P \) we have \( f(\xi) \geq 0 \), and that \( A \xi \) lies in the interior of \( P \) for some \( \xi \in X \). Then there exists \( \mu \in P^* \) such that \( f(\xi) = \mu(A \xi) \) for all \( \xi \in X \).

While our version applies only to the case \( Z = C(Y) \), our hypotheses (a) on \( f \) and \( A \) are somewhat weaker than those imposed by Luenberger—they are necessary as well as sufficient for (b) the existence of \( \mu \).

To understand our overall strategy of proof, consider the linear map \( (f, A) : X \to \mathbb{R} \times C(Y) \). As we will see below, (a) implies that the image of \( (f, A) \) avoids the interior of the orthant \( \mathbb{R}^+ \times P \).

To gain some intuition, consider the dual space \( X^* \) and the adjoint map \( (f, A)^* \). We can rephrase (b) to say that some vector in the kernel of \( (f, A)^* \) is in \( \mathbb{R}^+ \times P \). When \( Y \) has finite cardinality \( n \), we put the standard Euclidean inner product on \( \mathbb{R} \times C(Y) \cong \mathbb{R}^{n+1} \), and identify this space with its dual. Then the kernel of \( (f, A)^* \) and the image of \( (f, A) \) are orthogonal complements in \( \mathbb{R}^{n+1} \). The standard Farkas alternative (see Figure 2) says that, given any closed orthant \( O \), it must intersect one out of any pair of orthogonal complements. Our argument in the general case when \( Y \) might be infinite will be guided by this intuition.

**Proof of Theorem 5.4.** One direction is easy: suppose we have a positive Radon measure \( \mu \) so that for each \( \xi \in X \),

\[
f(\xi) = \int_Y A \xi \, d\mu.
\]

For any \( \xi \in X \) with \( f(\xi) = -1 \), write \( z = A \xi \in C(Y) \). We have \( \int z \, d\mu = -1 \), and since \( \mu \) is a positive measure, we can replace the function \( z \) with its negative part to conclude that \( \int z^- \, d\mu \geq 1 \). We also know \( \mu \) has finite total mass \( \text{mass}(\mu) := \int d\mu < \infty \). Therefore

\[
\text{Dist}(z, P) = \|z^-\| \geq 1/\text{mass}(\mu) > 0.
\]

This completes the proof that (b) implies (a).

To prove the converse, first give \( \mathbb{R} \times C(Y) \) the Euclidean combination of the sup norms on \( \mathbb{R} \) and \( C(Y) \):

\[
\|(a, g)\| = \sqrt{a^2 + \|g\|^2}.
\]

Now consider the orthant \( O := (1, \infty) \times P \). Our hypothesis (a) implies that there is positive distance between \( O \) and the image \( I := (f, A)(X) \) of the linear map \( (f, A) \). Take sequences \( (f(\xi_i), A \xi_i) \) in \( I \) and \( (t_i, z_i) \) in \( O \), whose pairwise distance approaches \( \text{Dist}(I, O) \). That is, setting

\[
v_i := (t_i - f(\xi_i), z_i - A \xi_i)
\]

we have \( \|v_i\| \to \text{Dist}(I, O) \).

If we were in an inner product space, and could extract a convergent subsequence of the \( v_i \), its limit would be orthogonal to \( I \), and writing the limit as \( (-c, c\mu) \), our only task would be to prove that this was in \( \mathbb{R}^+ \times P \) so that \( \mu \in P \) as desired. In our more general setting, we begin by replacing the \( v_i \) by vectors in \( \mathbb{R}^+ \times P \), and defer convergence and an analogue of orthogonality until later in the proof.

So our first claim is we can assume that \( v_i \in \mathbb{R}^+ \times P \). Certainly we can assume \( z_i = (A \xi_i)^* \), since this positive part of the function \( A \xi_i \) realizes \( \|z_i - A \xi_i\| = \|A \xi_i^+\| = \text{Dist}(A \xi_i, P) \). Then \( z_i - A \xi_i = -A \xi_i^- \in P \). Similarly, \( \text{Dist}(f(\xi_i), -1, \infty) = 1/1 - f(\xi_i) \) if it is positive (and zero otherwise), so we may assume \( t_i = \min(1, -1, f(\xi_i)) \). Thus \( t_i \leq f(\xi_i) \), so \( t_i - f(\xi_i) \leq 0 \). This proves the first claim.

We now have a geometric problem: from Figure 2 we see there is a special case where both the image of \( (f, A) \) and its orthogonal complement lie on the boundary of \( \mathbb{R}^+ \times P \). If this happens, then the closure \( \overline{I} \) intersects either the subspace \( \mathbb{R} \times \{0\} \) or \( \{0\} \times P \). The second case does not trouble us, but the first would cause us problems later; we now show that our assumption (a) rules it out. To do so, we think about the setup above geometrically—if \( \overline{I} \) intersects \( \mathbb{R} \times \{0\} \), then we expect that \( t_i - f(\xi_i) \to 0 \).

Thus our second claim is that we can assume the \( t_i - f(\xi_i) \) are uniformly negative. If not, \( lim f(\xi_i) \leq 1, \) so without loss of generality, we can rescale \( \xi_i \) down so that \( f(\xi_i) = -1 \). That means \( (-1, A \xi_i) \in I \). By hypothesis (a) we know

\[
d_i := \text{Dist}((-1, A \xi_i), O) = \text{Dist}(A \xi_i, P) \geq \varepsilon
\]

for some fixed \( \varepsilon > 0 \). Since we are using the Euclidean combination of the norms on \( \mathbb{R} \) and \( C(Y) \), the distance from any rescaling by \( k \) of \( (-1, A \xi_i) \) to \( O \) is given by the Euclidean distance from \( (-k, k\|A \xi_i^+\|) \) to \( (-1, 0) \). And we can use plane geometry to see that rescaling by \( 1 - \varepsilon^2 \) brings us closer to \( O \):

\[
\text{Dist}((-1 - \varepsilon^2)(-1, A \xi_i), O) \leq d_i \sqrt{1 - 2\varepsilon^2 + \varepsilon^4 + \varepsilon^4/d_i^2} \\
\leq d_i \sqrt{1 - \varepsilon^2 + \varepsilon^4}.
\]

Assuming, as we may, that \( \varepsilon < 1 \), the constant \( \sqrt{1 - \varepsilon^2 + \varepsilon^4} \) is less than 1. Therefore

\[
\text{Dist}(I, O) \leq \lim \text{Dist}((-1 - \varepsilon^2)(-1, A \xi_i), O) \\
< \lim \text{Dist}((-1, A \xi_i), O) \\
= \text{Dist}(I, O).
\]

This contradiction proves the second claim.

We have proved that the \( v_i \) are in \( \mathbb{R}^+ \times P \). We must now use analogues of orthogonality and completeness to construct a limit of the \( v_i \) and show it would lie in the kernel of \( (f, A)^* \). In an inner product space, the linear functional defined by a
vector $v$ vanishes on the orthogonal complement of $v$. Without an inner product, we can use this property to define a kind of orthogonality. Using the Hahn-Banach theorem, for each $i$ we can find a linear functional $(c_i, \nu_i) \in \mathbb{R} \times C^*(Y)$ that vanishes on $I$, satisfies $(c_i, \nu_i)(v_i) = 1$, and has norm

$$
\| (c_i, \nu_i) \| = 1 / \text{Dist}(v_i + I, (0, 0)).
$$

Because the $\mathbb{R}$-components of $v_i$ are uniformly negative, so are the $c_i$.

Similarly, in a complete normed linear space, we could extract a convergent subsequence of the $v_i$, since their norms are bounded above. Without this property, we take advantage instead of Alaoglu’s theorem, which tells us that the dual space to $\mathbb{R} \times C(Y)$ has the required completeness even though $\mathbb{R} \times C(Y)$ does not. By Alaoglu’s theorem, the $(c_i, \nu_i)$ have a subsequence converging in the weak* topology to a limit functional $(c, \nu)$; its norm is at most $1 / \lim \text{Dist}(v_i, -I) = 1 / \text{Dist}(I, O)$ and we have $c < 0$.

Setting $\mu := \nu / |c| \in C^*(Y)$, we claim this will be the Radon measure in statement (b). By construction, $(-1, \mu)$ vanishes on $I$, meaning that for $\xi \in X$, we have

$$
-f(\xi) + \int_Y A\xi \, d\mu = 0.
$$

In particular, this means that our weakened idea of orthogonality was enough to conclude that $(-1, \mu)$ is in the kernel of $(f, A)^*$. (Notice that we have used the additional geometric information that $I$ does not approach $\mathbb{R} \times \{0\}$ in an essential way; if it had, then $c$ would have vanished, and we would have been unable to rescale $\nu$ by $1/|c|$ to obtain the equation above.)

It remains only to show that $\mu$ is positive. In an inner product space, this would be obvious: each $\nu_i$ would be positive (since it was dual to a positive $z_i - A\xi_i$), and $\nu$ would be a limit of positive measures. But our $\nu_i$ were constructed implicitly by the Hahn-Banach theorem, and so might include negative pieces. We now address this problem.

We can decompose each $\nu_i$ into its positive and negative parts $\nu_i = \nu_i^+ - \nu_i^-$, with $\text{mass}(\nu_i) = \text{mass}(\nu_i^+) + \text{mass}(\nu_i^-)$. To show $\nu$ is positive, we will prove $\lim \text{mass}(\nu_i^+) = \lim \text{mass}(\nu_i^-)$. By construction, we know that

$$
1 = (c_i, \nu_i)(v_i) = c_i(t_i - f(\xi_i)) + \int_Y z_i - A\xi_i \, dv_i.
$$

Since $z_i - A\xi_i \in P$, we have

$$
\int_Y z_i - A\xi_i \, dv_i \leq \int_Y z_i - A\xi_i \, dv_i^+ \leq \| z_i - A\xi_i \| \text{mass}(\nu_i^+).
$$

Using Cauchy-Schwarz, and the two equations above, we get

$$
1 \leq \| v_i \| \sqrt{|c_i|^2 + (\text{mass}(\nu_i^+))^2}.
$$

Now $\| v_i \|$ converges to $\text{Dist}(I, O)$, so we find that $\lim \| (c_i, \nu_i) \| \geq 1 / \text{Dist}(I, O)$. But the limit of $\| (c_i, \nu_i) \|$ (which cannot be smaller) equals $1 / \text{Dist}(I, O)$. Therefore, $\lim \text{mass}(\nu_i^+) = \lim \text{mass}(\nu_i)$, completing the proof.

To apply Theorem 5.4 to optimization problems, we will let $X$ be the space of variations $\xi$ of our given configuration and $Y$ be the set of active constraints. Then we let $f(\xi)$ and $A\xi(y)$ be the directional derivatives of the objective function and of the constraint $y$. In this context, Theorem 5.4 says that a configuration is (a) strongly critical if and only if it is (b) balanced.

Note that our strong criticality is indeed stronger than a simple criticality condition, which would say that whenever $f(\xi) = -1$ we have $\text{Dist}(A\xi, P) > 0$, or equivalently that no $\xi$ has $f(\xi) < 0$ but $A\xi \in P$.

**Example 5.6.** With $X = \mathbb{R}^2$ and $Y = [0, 1]$, we can set $f(x_1, x_2) = x_1$ and $A(x_1, x_2)(y) = 2x_1 \sqrt{y - y^2} + x_2 y$ to give an example that is critical, but not strongly critical (and not balanced).

However, when $Y$ is a finite set (with the discrete topology), strong criticality is equivalent to criticality. For suppose whenever $f(\xi) = -1$ we have $\text{Dist}(A\xi, P) > 0$, but there is no uniform lower bound $\varepsilon > 0$ on this distance. For each $y \in Y$, we know that $A\xi(y)$ is a linear functional on $\xi$. Since there are only finitely many $y$, the graph of $\min_{y \in Y} A\xi(y)$ describes a polyhedron in $X \times \mathbb{R}$. Since the supremum over $\xi \in X$ is finite (we know it is nonpositive), it is achieved (at some $\xi$ corresponding to a vertex of this polyhedron). But for any $\xi$, the value is negative, so this supremum must be negative.

This allows us to recover the finite-dimensional Kuhn–Tucker theorem: let $X$ be the tangent space to $\mathbb{R}^n$ at $p$, let $Y$ be the finite set of active constraints at $p$, and let $f$ and $A$ be the directional derivatives of the objective function and the active constraints. Because $Y$ is finite, (a) is equivalent to the definition of constrained criticality above, and we obtain Theorem 5.2.

### 6. The Balance Criterion for Gehring Ropelength

We now have all the tools we need to develop a balance criterion, characterizing critical configurations for the Gehring ropelength problem. We start with definitions of criticality, guided by our version of Kuhn–Tucker.

**Definition.** Suppose $L$ is a rectifiable link with $\text{GThi}(L) = \tau$, and consider the Gehring ropelength problem of minimizing length subject to the constraint that $\text{GThi} \geq \tau$.

We say $L$ is a local minimum for Gehring ropelength if for all $L'$ sufficiently $C^0$-close to $L$ with $\text{GThi}(L') \geq \tau$ we have $\text{Len}(L') \geq \text{Len}(L)$.

We say $L$ is critical if for all $\xi \in \text{VF}^\infty(L)$ with $\delta\xi \text{Len}(L) < 0$ we have $\delta\xi \text{GThi}(L) < 0$.

We say $L$ is strongly critical if there exists $\varepsilon > 0$ so that for all $\xi \in \text{VF}^\infty(L)$ with $\delta\xi \text{Len}(L) = -1$, we have $\delta\xi \text{GThi}(L) \leq -\varepsilon$.
With these definitions, we can now apply our Kuhn–Tucker theorem to the Gehring ropelength problem.

**Theorem 6.1 (Balance Criterion for Gehring ropelength).** A link \( L \) is strongly critical for length when constrained by Gehring thickness if and only if there exists a positive Radon measure \( \mu \) on \( \text{GStrut}(L) \) such that, for every smooth vector-field \( \xi \) along \( L \), we have

\[
\delta_\xi \text{Len}(L) = \int_{\text{GStrut}(L)} A_G \xi \, d\mu,
\]

where \( A_G = \delta \text{Dist} \) is the rigidity operator.

**Proof.** We will apply Theorem 5.4 with \( X := \text{VF}^\infty(L) \) and \( Y := \text{GStrut}(L) \), letting \( f = \delta \text{Len}(L) \) be the derivative of length and \( A = A_G \) be the rigidity operator. We have

\[
\| (A_G \xi)^- \| = - \min_{\text{GStrut}} \delta_\xi \text{Dist} \{ x, y \}
\]

(when this is nonnegative). By Corollary 3.2, the right-hand side is \( -\delta^G \text{GThi}(L) \), so that condition (a) from Theorem 5.4 is exactly strong criticality. \( \square \)

**Smoothness of critical curves.** It is unclear, a priori, how much regularity one should expect for ropelength-critical curves in the Gehring problem. But we can use the balance criterion to deduce immediately that they must have finite total curvature.

**Corollary 6.2.** If a link \( L \) is strongly critical for the Gehring ropelength problem, then \( L \) is FTC.

**Proof.** The theorem tells us that \( L \) can be balanced:

\[
\delta_\xi \text{Len}(L) = \int_{\text{GStrut}(L)} A_G \xi \, d\mu.
\]

But the right-hand side is a distribution of order zero on \( \xi \), since

\[
\int_{\text{GStrut}(L)} A_G \xi \, d\mu \leq \text{mass}(\mu) \sup \| \xi \|.
\]

Therefore, by Lemma 4.1, \( L \in \text{FTC} \). \( \square \)

We can now rewrite the conclusion of our balance criterion in terms of the curvature force on \( L \) and the adjoint \( A_G^- \) of the rigidity operator.

**Corollary 6.3.** A link \( L \) is strongly critical for Gehring ropelength if and only if it has finite total curvature and there exists a positive Radon measure \( \mu \) on \( \text{GStrut}(L) \) such that

\[
A_G^-(\mu) = -\mathcal{K}
\]

as forces along \( L \).

**Proof.** The theorem guarantees that for all smooth \( \xi \), we have

\[
\delta_\xi \text{Len}(L) = \int_{\text{GStrut}(L)} A_G \xi \, d\mu.
\]

By the corollary, \( L \) is FTC, so the left-hand side can be rewritten as \( -\int_L \langle \xi, d\mathcal{K} \rangle \). Approximating any continuous vector-field uniformly by smooth ones, we find that

\[
-\int_L \langle \xi, d\mathcal{K} \rangle = \int_{\text{GStrut}(L)} A_G \xi \, d\mu
\]

for all \( \xi \in \text{VF}(L) \), or in other words, \( -\mathcal{K} = A_G^-(\mu) \).

We get an immediate and useful geometric corollary to this balance criterion.

**Corollary 6.4.** Suppose \( L \) is a Gehring-critical link, and \( E \subset L \) is a subset with nonzero net curvature \( \mathcal{K}(E) \in \mathbb{R}^3 \). Then there must be at least one Gehring strut \( \{ e, x \} \) with \( e \in E \) and \( x \notin E \), and \( \mathcal{K}(E) \) is in the convex cone generated by the directions \( x-e \) of all such struts.

**Proof.** First note that struts from \( E \) to \( E \) contribute no net force. By the balance criterion, we have \( \mathcal{K}(E) = -A_G^-\mu(E) \), and the latter is a weighted sum of vectors \( x-e \).

We note that this corollary is the analogue for Gehring ropelength of von der Mosel and Schuricht’s “Characterization of Ideal Knots” [vdMS03, Thm. 1].

We next find that critical links are \( C^1 \) as well as FTC:

**Proposition 6.5.** If \( L \) is strongly critical for Gehring ropelength, then \( L \) is \( C^1 \).

**Proof.** We already know that \( L \) has finite total curvature; it is \( C^1 \) precisely when it has no corners, that is, when the curvature force \( \mathcal{K} \) has no atoms. If \( T_{\pm} \) are the right and left tangent vectors to \( L \) at \( x \), then \( \mathcal{K}(\{x\}) = T_+ + T_- \). When \( \mathcal{K}(\{x\}) \neq 0 \), Corollary 6.4 says there exists at least one strut \( \{x,y\} \) with \( \langle y-x, \mathcal{K}(\{x\}) \rangle > 0 \). That is,

\[
\langle y-x, T_+ \rangle + \langle y-x, T_- \rangle > 0,
\]

so we must have \( \langle y-x, T_+ \rangle > 0 \) or \( \langle y-x, T_- \rangle > 0 \). (See Figure 3.) In either case it follows that there exist points on \( L \) near \( x \) that are closer to \( y \) than \( x \) is, which contradicts the hypothesis that \( \{x,y\} \) was a Gehring strut. This completes the proof. \( \square \)

The example of the tight clasp in Section 9 shows that the curvature need not be bounded, but so far this is the worst behavior we can display. We conjecture that the curvature measure is always absolutely continuous with respect to arclength.

**The Mangasarian–Fromovitz constraint qualification.** Corollary 6.3 will be the basic model for balance criteria for generalized links, and for links constrained by other thickness functionals [CFK+04]. In some cases, including this one, we will be able to improve on this form of the criterion by replacing strong criticality with criticality. This is our next goal.

In Section 5, we introduced the idea of a regular or constraint-qualified point for a finite set of constraints \( g_1,\ldots,g_n \) such a point has some variation direction \( v \) such that \( D_v g_i > 0 \) for all the active \( g_i \). By Corollary 3.2, the corresponding idea for a link \( L \) in the Gehring ropelength problem
is the existence of a vector field $\xi$ for which $\delta^+_L \text{GThi}(L) > 0$. But this is automatic: dilating $L$ increases $\text{GThi}$ to first order.

This regularity for our problem allows us to prove that local minima are critical and that critical points are strongly critical.

**Proposition 6.6.** A link $L$ is critical for the Gehring ropelength problem if and only if it is strongly critical. If $L$ is a local minimum, then $L$ is critical.

**Proof.** Suppose $L$ is a local minimum but not critical. Then for some $\xi \in \text{VF}^\infty(L)$ we have $\delta^\xi \text{Len}(L) < 0$ but $\delta^+_L \text{GThi}(L) \geq 0$. Then for small enough $t > 0$, the link $L + t\xi$ has less length than $L$. This contradicts minimality unless $\delta^+_L \text{GThi}(L) = 0$ and thickness has decreased (but not to first order). But in this case, we can instead use a rescaling $(1 + Bt^2)(L + t\xi)$, for some constant $B > 0$. For small $t > 0$ these again have less length than $L$, contradicting minimality.

Strong criticality always implies criticality. Conversely, suppose a closed link $L$ is critical but not strongly critical. Then there exists a sequence $\xi_i \in \text{VF}(L)$ with $\delta^\xi_i \text{Len}(L) = -1$ and $\delta^+_L \text{GThi}(L) \to 0$. Let $\eta$ be the vector field along $L$ induced by dilation, scaled so that $\delta^\eta \text{Len}(L) < 1$. Then we observe that $\delta^\eta_{\eta+\xi_i} \text{Len}(L) < 0$ for all $i$. Corollary 3.3 shows that

$$\lim_{\eta+\xi_i} \delta^+_{\eta} \text{GThi} \geq \delta^+_\eta \text{GThi} > 0.$$ 

But then for some $i$, we must have

$$\delta^+_{\eta+\xi_i} \text{Len}(L) < 0, \quad \delta^+_{\eta+\xi_i} \text{GThi}(L) > 0,$$

contradicting the criticality of $L$. 

Thus for closed links, a minimizer (or more generally any critical point) for the Gehring ropelength problem is strongly critical, and hence by Corollary 6.3 its curvature force is balanced by some strut force $A^*_\text{G} \mu$. However, in our generalized ropelength problems, with endpoint constraints and obstacles, constraint qualification will not always hold. Then we will have to be careful about the distinction between criticality and strong criticality.

**Existence of minimizers.** We now show that each link-homotopy class contains a globally length-minimizing curve with $\text{GThi}(L) \geq 1$.

**Proposition 6.7.** In a given link-homotopy class $[[L]]$, among all curves with Gehring thickness at least 1, there is some $L_0$ of minimum length.

**Proof.** We can rescale the initial $L$ so that it has $\text{GThi}(L) \geq 1$. Thus, the set over which we are minimizing length is nonempty. We can assume that $L$ is rectifiable as follows: for any $x \in L$, consider $L$ intersected with the ball of radius $1/4$ around $x$. One component of this intersection is an arc containing $x$ and this can be replaced by a straight line.

Let $L_1, L_2, \ldots$ be a length-minimizing sequence with $\text{GThi}(L_i) \geq 1$ and $\text{Len}(L_i) \leq \text{Len}(L)$. This uniform length bound means that we can apply Arzela–Ascoli to find a subsequence converging in $C^0$ to a limit $L_0$. But length is lower semicontinuous, and $\text{GThi}$ is continuous under $C^0$ convergence. So $\text{GThi}(L_0) \geq 1$ and $L_0$ minimizes length.

It remains only to check that $L_0 \in [[L]]$. But if $i$ is large enough, the $C^0$ distance from $L_0$ to $L_i$ is less than $1/2$. Then no two components intersect during the straight-line homotopy from $L_i$ to $L_0$. So $L_0$ and $L_i$ are link-homotopic, as desired.

This global minimum for length will certainly be a local minimum, and thus a critical point for the Gehring ropelength problem. As we have discussed, this implies that it is $C^1$ and has finite total curvature.

It is conventional in knot theory to restrict one’s attention to link types that are *tame*, meaning isotopic to a finite polygon (or, equivalently, to a $C^\infty$ curve). We have certainly restricted our attention to links of finitely many components, but it is natural to wonder whether we have to worry about wild links for our Gehring problem. In fact, we do not:

**Proposition 6.8.** There are no wild link-homotopy types with finitely many components.

This result follows easily from a lemma of Milnor [Mil54] that any link has arbitrarily close homotopic polygonal approximations. We give an alternate proof using our results on Gehring minimizers.

**Proof.** Such a link would be link-homotopic to a Gehring minimizer by Proposition 6.7. This minimizer would be FTC by Corollary 6.2. And Milnor showed [Mil50] that every link of finite total curvature is tame.

Thus we need consider only tame links in the work to come.

### 7. EXAMPLES OF GEHRING-CRITICAL LINKS

**Known length-minimizing links.** In [CKS02], we showed that if one component of a link is linked to $k$ others then its length is at least a certain constant $P_k$. Although our theorem was written for the original ropelength problem, the proof is valid for Gehring thickness as well. Whenever a link can be
realized with each component having length $P_k$, that configuration is thus a length-minimizer for both the original ropelength problem and the Gehring problem. (These are still the only examples known to be ropelength-minimizing.)

To any link $L$ we can associate a graph: the vertices are the components of $L$, and the edges record which pairs are nontrivially linked. For any tree $T$ with $n$ edges, there is a unique link $H(T)$ that is a connect sum of $n$ Hopf links and whose associated graph is $T$.

For many trees $T$ with vertices of sufficiently low degree, we can realize $[H(T)]$ explicitly with each component having exactly its minimum possible length $P_n$. Even some slightly more complicated links, like the example in Figure 4, can be realized in this way. The distance between any two linked components is exactly 1. Each component in one of these minimizers is a convex plane curve built from circular arcs of radius 1 and straight segments. It is an outer parallel (at distance 1/2) to a shortest curve surrounding its components. Unlike the previous situation, however, these measures result in perimeter-minimizing enclosures of $n = 1, 2, 3$ and 4 unit-diameter disks in the plane. The components in the known minimizing links are outer parallels to such curves at distance 1/2. When $n = 4$, the minimizer does not have a unique shape; instead there is a one-parameter family of minimizers.

**Example 7.1.** The link $H(T_2)$ is the simple chain of three components, shown in Figure 6. In the ropelength minimizer, the two end components are circles $C_1$ and $C_2$, while the middle component is a stadium curve $S$. The centers of the circular arcs in $S$ are points $c_i \in C_i$, while the center of $C_i$ is a point $s_i \in S$. The Gehring struts are exactly where different components are at distance 1. There is a strut from each point along each circular arc to the center of that arc (from $C_i$ to $s_i$ and from $S$ to $c_i$). There is also one further strut $\{c_1, c_2\}$.

Since we know this configuration is length-minimizing when constrained by Gehring thickness, these struts, by Corollary 6.3 and Proposition 6.6, must support a balancing measure $\mu$. Conversely, exhibiting such a measure will reprove that this configuration is critical for the Gehring ropelength problem, though to re-prove it is a local minimum would require some second-order theory. We now provide such a measure, which will be a useful comparison of the results of this paper against the results of [CKS02].

Except for $c_i$, each point $x$ along the component $C_i$ is part of a unique strut $\{x, s_i\}$. The measure assigned to struts in this “wheel” must exactly balance the curvature force $dK = N ds(x)$ along $C_i$. Because the wheel forms a complete circle, at the center points $s_i$, the incoming forces from these struts cancel one another, leaving no net force.

The situation on the stadium curve is slightly more complex. The struts from the semicircles of $S$ to the points $c_i$ again balance $dK = N ds(x)$, now for $x$ along the semicircles. Unlike the previous situation, however, these measures...
have a resultant inward force of magnitude 2 at $c_i$, directed parallel to the straight segments in the stadium curve. To balance these forces, the measure $\mu$ must have an atom of magnitude 2 at the one remaining Gehring strut $\{c_1, c_2\}$.

The measure $\mu$ we have described does balance the curvature force everywhere along the link, and thus demonstrates that the link is critical for Gehring ropelength.

It is worth emphasizing the fact that the inner Gehring strut $\{c_1, c_2\}$ bears an atom of $\mu$. This stresses the point that in our Kuhn–Tucker theorem and the resulting balance criterion we are required to view the Lagrange multiplier $\mu$ as a Radon measure in the dual space $C^*(G\text{Strut}(L))$, rather than as a density function on struts.

Although balanced and ropelength-minimizing, Example 7.1 is not rigid, in the sense that the components $C_i$ can be pivoted around the points $c_i$ to be centered at any points $s_i$ on the semicircles of $S$.

A stronger form of nonuniqueness is exhibited by the minimizing configurations [CKS02] of the five-component link $H(T_4)$, with one component linked to all four others. Here the central component does not even have a uniquely determined shape. Instead there is a one-parameter family of minimizing shapes, corresponding to the deformation seen in Figure 5 for $n = 4$. Again, each of the minimizers can be balanced. (As we have proven, the existence of the balancing measure $\mu$ is equivalent to strong criticality for the ropelength problem, but it does not imply that the critical point is isolated.)

For $n > 5$, we expect that similar configurations of $H(T_n)$, like the one shown in Figure 7 for $n = 6$, are again minimizing. Our balance criterion lets us show they are at least critical:

**Proposition 7.2.** Suppose $P$ is a convex planar $n$-gon with unit-length sides and exterior angles in $[0, 2\pi/3]$. Let $L_0$ be the outer parallel at distance 1 to $P$, and let $L_1, \ldots, L_n$ be unit circles, perpendicular to the plane of $P$, passing through the vertices of $P$, and centered at points on $L_0$. Then the link $L = L_0 \cup L_1 \cup \cdots \cup L_n$ (a configuration of $H(T_n)$ with Gehring thickness 1) is critical for Gehring ropelength.

**Proof.** As in the simple chain, each circle $L_i$ focuses a wheel of struts to its center point on $L_0$, and a measure assigning force $\delta s_i$ to these struts balances the curvature force on each circle while exerting no net force on $L_0$.

Let $c_i$ be the vertex of $P$ on $L_i$, and let $2\alpha_i$ be the exterior angle of $P$ there. The condition $\alpha_i \leq \pi/3$ exactly suffices to know that no two vertices (and thus no two $L_i$) are at distance less than 1 from each other, confirming that $G\text{Th}(L) = 1$. The curve $L_0$ includes an arc of the unit circle around $c_i$; from this arc of length $2\alpha_i$ a fan of struts converge to $c_i$. To balance the curvature force on $L_0$, these struts again have measure equal to $\delta s_i$, giving a net inward force of $2\sin \alpha_i$ on $c_i$. The remaining, isolated struts of $L$ connect successive $c_i$ along the edges of $P$. Unit atoms of compressive force on these isolated struts produce exactly the outward forces $2\sin \alpha_i$ at $c_i$ needed to balance the inward forces from $L_0$.

By Corollary 6.3, the existence of this balancing measure on the Gehring struts proves that $L$ is critical. \hfill $\square$

For $n \leq 5$, we know these configurations for $H(T_n)$ are ropelength minimizers. For $n > 5$, the component $L_0$, having length $n + 2\pi$, is longer than it needs to be: at the expense of lengthening some other components, it could be shortened to length $P_n$. Asymptotically, this is much smaller, being $O(\sqrt{n})$. However, calculations we have done suggest that the critical configuration described above is the global minimum for ropelength.

The examples given in Proposition 7.2—their critical configurations and presumed minimizers for $H(T_n)$—are quite interesting. The shape of $L_0$ is free to move in a $(n - 3)$-parameter family; each other component is free to pivot (about its vertex of $P$ and along one of the arcs of $L_0$), giving an additional $n$ parameters for the shape of the whole link $L$. We also note that these examples are tight links that are not packed tightly: Consider the thick tube around one of these configurations. As $n$ increases, it occupies an ever-smaller fraction of the volume of its convex hull. This should be compared with experiments of Millett and Rawdon [MR03] on this volume fraction.

Although we have stated Proposition 7.2 above only for stars $T_n$, the same balancing works for the links $H(T)$ based on other trees $T$. Each component linked to $n$ others should have the shape of $L_0$ above. We note, however, that critical links built in this way are not always minimizers.

**Example 7.3.** Consider the tree $T_{n,m}$, with $n + m$ vertices, including one of valence $n$ connected to another of valence $m$. If $H(T_{n,m})$ is built according to the recipe above, with the two long components touching, the total length is $(2\pi + 1)(n + m)$. However, when $n$ and $m$ are large (at least 12) we can save two units of length by letting these two components move apart from each other into a loose Hopf link. Each of them is tightly wrapped by $n - 1$ or $m - 1$ short components, but the two sets don’t touch each other at all and are instead free to move
independently. Again, the force balancing proceeds just as in Proposition 7.2. These critical configurations (which we presume are the ropelength minimizers) for \( H(T_{n,m}) \) are the first known in which certain pairs of linked components are not in contact.

In all of the examples \( H(T) \) discussed above, each component is a planar curve built from straight segments and arcs of unit circles. The proven minimizers are minimizers in their isotopy class for the original ropelength problem, as well as minimizers in their link-homotopy class for Gehring ropelength. In fact, in [CFK+04] we will consider a family of thicknesses with varying stiffness. Each of these thicknesses is characterized by a lower bound \( \lambda \) on the diameter of curvature. These \( H(T) \) are ropelength-minimizers for the whole family, as long as the stiffness \( \lambda \) does not exceed 2, when circular arcs of larger radius would be needed. We will also develop an analog of our balance criterion for these other ropelength problems, and will see that all the \( H(T) \) discussed above (including those that are not minimizers) are critical for all formulations of ropelength.

**Nonembedded Gehring-critical links.** To illustrate the differences between the Gehring problem and the original ropelength problem, we now give some examples of a different flavor, critical configurations that are nonembedded and thus have infinite ropelength in the original sense.

Any knot is of course link-homotopic to the unknot. The Gehring ropelength minimizer degenerates to a point (of length zero). The same happens for any component of an arbitrary link that is link-homotopically split from the rest of the link.

Milnor showed that, up to link homotopy, links of two components are classified by their linking number [Mil54]. When the linking number is zero, the components split, and the Gehring minimizer degenerates to have length zero. We can, however, also describe an unstable critical configuration for this unlink: one component degenerates to a point \( p \) while the second is a unit circle centered at \( p \).

The case of linking number 1 is close to Gehring’s original problem: the minimizer is the same Hopf link built from round circles. (This case fits in the class \( H(T_n) \) considered above.) For larger linking number, we can use Corollary 6.3 to exhibit many Gehring-critical configurations.

**Example 7.4.** For linking number \( mn \) there is a critical configuration \( L_{m,n} \) consisting of the minimizing Hopf link, with one component covered \( m \) times and the other \( n \) times. Its total length is thus \( 2\pi(m+n) \). There are other critical configurations, sometimes shorter. For example, each component can be a figure-eight built from two tangent circles. Figure 8 shows a configuration like this with total length \( 2\pi(m+n) \) and linking number \( mn - m_1n_1 \). The best configurations we know for linking number 17, for instance, use \((m,n) = (6,3)\) or \((4,5)\). Assuming configurations like these are the minimizers for two-component links, they give examples where the set of minimizers is disconnected (since we can interchange the two components, or reorder the way one component covers its figure-eight).

None of these configurations is embedded, so they are not critical points for the original ropelength problem: as expected, the extra freedom in the Gehring problem sometimes allows for shorter solutions. As a further example, consider the \((2,4)\)-torus link, with linking number 2. We have computed the presumed ropelength-minimizer numerically, as in [Sul02]. The results are shown in Figure 9; this solution is longer than the covered Hopf link \( L_{2,1} \) (the presumed Gehring-minimizer) and is not even critical for the Gehring problem.

For links of more than two components, linking numbers do not suffice to distinguish link-homotopy types; we must also consider Milnor’s \( \mu \)-invariants [Mil54, HL90]. For instance, the Borromean rings, with no nonzero linking numbers, belong to a nontrivial link-homotopy class because they have \( \mu \)-invariant equal to 1.

Numerical experiments in Brakke’s Evolver (compare [Sul02]) suggest that the minimizing Borromean rings for the Gehring-ropelength problem should consist of three congruent curves in perpendicular planes. In [CKS02], we described such a configuration built from circular arcs

---

**Figure 8:** In this configuration of two curves from Example 7.4, each circle is covered \( m_i \) or \( n_i \) times, as labeled. If \( m = m_1 + m_2 \) and \( n = n_1 + n_2 \), then the curve has total length \( 2\pi(m+n) \), and linking number \( mn - m_1n_1 \). It is constrained-critical, though often not minimal, for the Gehring problem in the link-homotopy class defined by its linking number.

**Figure 9:** This picture shows a numerically computed minimizer for the original ropelength problem on the \((2,4)\)-torus link. Because it has a strut between two points on the same component (shown center, where the darker tube contacts itself), it is not balanced for the Gehring-ropelength problem considered here. It is longer than \( L_{2,1} \), the Hopf link with one component doubly covered, which we conjecture is the Gehring-minimizer for this link-homotopy type. Notice that both of these configurations break the symmetry between the components of the link, so we expect a (longer) critical configuration where the two components are congruent. (See [CFK+04] for more details on such applications of the idea of symmetric criticality.)
of radius 1. Unfortunately, Corollary 6.3 shows this configuration is not even length-critical when constrained by Gehring-thickness. In Section 10, the culmination of our paper, we will explicitly describe a very similar configuration of the Borromean rings, which we prove is critical and believe is the minimizer.

However, in order to solve for these Borromean rings, we must first consider a simpler interaction between two ropes: the clasp that occurs when one rope is pulled over another. Describing this will require a theory of generalized links.

8. GENERALIZED LINK CLASSES

Although some of our definitions have applied to arbitrary curves, so far we have been treating only ordinary (closed) links. We now want to consider generalized problems involving curves with endpoints. To get meaningful link classes in this setting, we must include constraints for the endpoints and obstacles for the link.

**Definition.** A generalized link \( L \) is a curve \( L \) (with disjoint components) together with obstacles and endpoint constraints. In particular, each endpoint \( x \in \partial L \) is constrained to stay on some affine subspace \( M_x \subset \mathbb{R}^3 \), which can have dimension 0, 1 or 2. Furthermore, there is a finite collection of obstacles for the link. Each obstacle
\[
\{ p \in \mathbb{R}^3 : g_j(p) < 0 \}
\]
is given by a \( C^1 \) function \( g_j \) with 0 as a regular value. By calling these obstacles, we mean that \( L \) is constrained to stay in the region where \( \min g_j \geq 0 \).

While we could allow even more general endpoint and obstacle constraints, this version fits nicely with our overall setup, and allows for all the specific examples we have in mind.

**Definition.** Given a generalized link \( L \), with obstacles \( g_j \) and endpoint constraints \( M_x \), its link-homotopy class \([L]\) is the set of all links \( L' \) that are link homotopic to \( L \) through links that avoid the obstacles and maintain the endpoint constraints. (As before, in a link homotopy, each component of \( L \) can intersect itself but not the others.)

This definition is comparable to our previous definition for closed links (page 2); as in the discussion at the end of Section 6, we may restrict our attention to tame link classes.

Given a generalized link \( L \), only variations preserving the endpoint constraints should be allowed. A vectorfield \( \xi \in VF(L) \) is said to be compatible with the constraints if it is tangent to \( M_x \) at each endpoint \( x \in \partial L \). We write \( VF_c(L) \) for the space of all compatible vectorfields.

Given a set of obstacles \( g_j < 0 \) and a link \( L \), we write
\[
O(L) := \min_j \min_{x \in L} g_j(x).
\]
Then \( L \) avoids the obstacles \( g_j \) if and only if \( O(L) \geq 0 \). We define the wall struts of \( L \) by
\[
Wall_j(L) := L \cap \{ g_j = 0 \}, \quad Wall(L) = \bigcup_j Wall_j(L).
\]
This reflects those parts of \( L \) on the boundary of the obstacle, but is not strictly speaking a subset of \( L \) since one point \( x \in L \) might be in several of the \( Wall_j \). When \( O(L) = 0 \), by Clarke’s Theorem 3.1 we have
\[
\delta^+ \xi O(L) = \min_j \min_{x \in Wall_j(L)} \langle \xi_x, \nabla_x g_j \rangle.
\]
Again, we collect the various derivatives appearing on the right-hand side into a rigidity operator \( AW : VF_c(L) \rightarrow C(Wall(L)) \) on wall struts, given by \( AW \xi := \langle \xi_x, \nabla g_j \rangle \). Its adjoint \( AW^* \) is then
\[
\int_L \xi \, dA_W(\mu) = \int_{Wall(L)} A_W \xi \, d\mu = \sum_j \int_{x \in Wall_j(L)} \langle \xi_x, \nabla x g_j \rangle \, d\mu(x).
\]
We also have corresponding definitions for locally minimal, strongly critical, and critical configurations of \( L \):

**Definition.** We say that a generalized link \( L \) is a local minimum for length when constrained by GThi if we have \( \text{Len}(L') \geq \text{Len}(L) \), for all sufficiently \( C^0 \)-close links \( L' \) with the same obstacle and endpoint constraints and with GThi\((L') \geq \text{GThi}(L) \). We say \( L \) is strongly critical (respectively, is critical) for minimizing length when constrained by GThi if there is \( \varepsilon > 0 \) such that for all compatible smooth \( \xi \) with \( \delta^+ \xi \text{Len} = -1 \), the quantity
\[
\min \left( \delta^+ \xi \text{GThi}(L), \delta^+ \xi O(L) \right)
\]
is at most \(-\varepsilon \) (respectively, is negative).

As in our discussion of Kuhn–Tucker, these notions will be equivalent only under a regularity assumption based on the constraint qualification of Mangasarian and Fromovitz [MF67]:

**Definition.** A generalized link \( L \) is GThi-regular if there is a thickening field, meaning a smooth compatible \( \eta \) for which \( \delta^+ \eta \text{GThi}(L) > 0 \) and \( \delta^+ \eta O(L) > 0 \).

Note that, while we require \( \eta \) to strictly increase GThi and to move away from the obstacle, both to first order, there is no corresponding requirement for the endpoint constraints, since they are linear equality constraints instead of nonlinear inequality constraints.

We can now prove a generalization of Proposition 6.6:

**Proposition 8.1.** If a generalized link \( L \) is a GThi-regular local minimum when constrained by GThi, then \( L \) is critical. Also, if \( L \) is GThi-regular and critical when constrained by GThi, then it is strongly critical.

**Proof.** The regularity of \( L \) means there exists a thickening field \( \eta \in VF_c(L) \). We may assume \( \delta^+ \eta \text{Len}(L) > 0 \) for otherwise \( L \) is neither minimal nor critical; we then scale \( \eta \) so that \( \delta^+ \eta \text{Len}(L) < 1 \).

Suppose \( L \) is a local minimum but not critical. Then for some compatible \( \xi \) we have \( \delta^+ \xi \text{Len}(L) < 0 \) while
\[ \delta_\xi^+ \text{GThi}(L) \geq 0 \text{ and } \delta_\xi^- O(L) \geq 0. \] For small \( t > 0 \), consider the links \( L_t = L + t(\xi + \eta) \). Then
\[ \frac{d \text{Len}(L_t)}{dt} \bigg|_{t=0} = \delta_\xi \text{Len}(L) + \epsilon \delta_\eta \text{Len}(L). \]

We choose \( 0 < \epsilon < -\delta_\xi \text{Len}(L)/\delta_\eta \text{Len}(L) \), so this derivative is negative at time 0. Thus for small \( t \), the \( L_t \) have length less than \( \text{Len}(L) \), contradicting minimality if they obey our constraints. But
\[ \frac{d \text{GThi}(L_t)}{dt} \bigg|_{t=0} > 0, \quad \frac{d O(L_t)}{dt^+} \bigg|_{t=0} > 0, \]
and the endpoint constraints are linear, so the links \( L_t \) meet all our constraints for small \( t > 0 \).

Now suppose \( L \) is critical but not strongly critical. Then there exists a sequence of compatible vectorfields \( \xi_i \in \text{VF}_c(L) \) with \( \delta_\xi \text{Len}(L) = -1 \) but with either \( \delta_\xi^+ \text{GThi}(L) \to 0 \) or \( \delta_\xi^- O(L) \to 0 \). Then we observe that \( \delta_{\eta+i\epsilon, \xi_i} \text{Len}(L) < 0 \) for all \( i \), while by Corollary 3.3 either
\[ \lim_{\eta+i\epsilon, \xi_i} \text{GThi} \geq \delta_\eta \text{GThi} > 0 \]
or
\[ \lim_{\eta+i\epsilon, \xi_i} O \geq \delta_\eta O > 0. \]
Taking \( i \) large enough that one of these quantities is already positive, we get a contradiction to the criticality of \( L \).

So far, this development has paralleled that of Section 6; we now diverge from our previous course. Earlier, we saw that every closed link is GThi-regular: rescaling always provides a thickening field. In the generalized setting, this is no longer the case. Thus minimality no longer implies criticality.

**Example 8.2.** To give a specific example, rotate the constraints of Example 5.1 around the \( z \)-axis to give obstacles
\[ g_1 = (x^2 + y^2 - 1)^3 - z < 0 \text{ and } g_2 = z < 0 \text{ for an unknot } L. \] The unit circle in the \( xy \)-plane is on the boundary of both obstacles, and is clearly the minimum length configuration in its homotopy class. However, it is not critical: shrinking it toward the origin will reduce its length to first order; the constraint \( g_1 \geq 0 \) is now violated, but not to first order.

Further, criticality and strong criticality may be different: if we allowed infinitely many obstacles, we could construct critical, but not strongly critical links by following the lead of Example 5.6. (If we do not allow infinitely many obstacles, then an open question remains: is strong criticality a stronger assertion than criticality?)

**Example 8.3.** To justify our emphasis on strong criticality (rather than restricting our attention to regular, critical links) we also note that it is easy to construct strongly critical links that are not regular; simply take \( g_1(x, y, z) = x^2 + y^2 - 1 \) (so the excluded region is the infinite cylinder around the \( z \)-axis), \( g_2 = -g_1 \), and \( L \) to be the unit circle in the \( xy \)-plane. This link is trapped on the cylinder \( g_1 = 0 = g_2 \), so it has no thickening field. On the other hand, it is clearly strongly critical.

Now we are ready to extend our Gehring balance theorem to the generalized setting. We will accommodate the endpoint constraints by restricting our attention to compatible vector-fields. Our other constraints are then \( \text{Dist} \geq 1 \) on \( L^{(2)} \) and \( g_j \geq 0 \) along \( L \). The set \( Y \) of active constraints or struts then consists of the Gehring struts together with the wall struts.

**Theorem 8.4.** A generalized link \( L \) is strongly critical for Gehring ropelength if and only if there is a positive Radon measure \( \mu \) on \( \text{GStrut}(L) \cup \text{Wall}(L) \), such that
\[ -K = (A_G + A_W)^* \mu \]
as linear functionals on \( \text{VF}_c(L) \). This means that \(-K\) and \((A_G\oplus A_W)^*\mu\) agree as forces along \( L \) except at endpoints \( x \in \partial L \), where they may differ by an atomic force in the direction normal to \( M_x \).

**Proof.** This is again a straightforward application of our Theorem 5.4. We use
\[ X = \text{VF}_c(L), \quad Y = \text{GStrut}(L) \cup \text{Wall}(L). \]
In Theorem 5.4 we then use \( f = \delta \text{Len} \) and \( A = A_G \oplus A_W \), the direct sum of the rigidity operators on Gehring struts and wall struts.

**Remark 8.5.** We can ignore the endpoints \( x \in \partial L \) when applying this theorem, as long as the link \( L \) meets each endpoint constraint \( M_x \) normally. We know of no examples of Gehring-critical generalized links where this is not the case.

To understand the interplay between Gehring and wall struts, we now offer a simple example of a Gehring-balanced generalized link \( L \) with nonempty \( \partial L \) needing nonzero force on the wall struts.

**Example 8.6.** Cut the simple chain of Figure 6 by parallel planes through \( s_1 \) and \( s_2 \) with normal vector \( c_1 - c_2 \), and let \( L \) be the part of the chain lying between the two planes. This generalized link includes two semicircles with endpoints normal to the planes, and also the inner stadium curve, which is tangent to the planes at \( s_1 \) and \( s_2 \). We let the planes bound an obstacle, forcing \( L \) to stay between the planes, and we use them also as endpoint constraints. Then \( L \) is Gehring-balanced: though the semicircles now exert a net outward force on \( s_1 \) and \( s_2 \), this is balanced by wall struts at these points. And the internal balance for the stadium curve remains the same.

9. THE TIGHT GEHRING CLASP

The tight configurations of Section 7 were the simplest closed links we could imagine: the Hopf link, and various connect sums of Hopf links in which each component is still a convex plane curve. But there is an even simpler interaction between two ropes, the clasp formed when one rope is pulled taut over another, as at the junctions of a woven net, or when a bucket is lifted from a well by passing a rope through its rope.
handle. We can model a single clasp as a generalized link with endpoint constraints.

To define the simple clasp, fix two parallel planes $P$ and $\tilde{P}$ at least 2 units apart. Then take two unknotted arcs $\gamma$ and $\tilde{\gamma}$ that lie between the planes, with the endpoints of $\gamma$ constrained to lie in $P$ and those of $\tilde{\gamma}$ in $\tilde{P}$. Let the complement of the slab between $P$ and $\tilde{P}$ be an obstacle for the generalized link $\gamma \cup \tilde{\gamma}$, and select the isotopy class of such links shown in Figure 10. This is the class where closing each arc in the plane of its endpoints would produce a Hopf link.

It is natural to assume that the minimizing configuration for this problem would consist of semicircular arcs passing through each others’ centers, together with straight segments for this problem would consist of semicircular arcs passing of its endpoints would produce a Hopf link. Figure 10. This is the class where closing each arc in the plane (see [CDFHT01, CS03]) is 2×2. Algebraically it is isomorphic to $D_4$.

To derive the Gehring-critical clasps can easily be extended to show these are the unique critical configurations among curves with this 2×2 symmetry. We omit the details, however, because we know of no way to show that the overall minimizers must have this symmetry. If one could prove this, it would then follow that our clasps are the minimizers.

A convenient parametrization. Our symmetry assumptions mean that the clasp is described by the shape of half of the component $\gamma$, from its tip along the $z$-axis into the $x>0$ half-plane and up to the plane $\tilde{P}$. This consists of a curved arc near the tip joined to a straight segment near $P$. Since the curved arc is strictly convex, we can parametrize it by the angle $\phi$ made by its tangent vector above the horizontal, as in Figure 12. In fact, we will use the sine of this angle, $u = \sin \phi$, as our parameter. Thus in the simple clasp, for $u \in [0, 1]$ we write

$$\gamma(\pm u) = (\pm x(u), 0, z(u)), \quad \tilde{\gamma}(\pm u) = (0, \pm x(u), -z(u)).$$

Elementary calculations show the following:

Lemma 9.2. For a convex curve $\gamma$ in the $xz$-plane, parameterized by the sine $u$ of its direction $\varphi \in [-\pi/2, \pi/2]$, the arclength $s$ satisfies

$$\frac{ds}{dx} = \frac{du}{\kappa \sqrt{1 - u^2}},$$

where the curvature $\kappa$ is given by

$$\kappa = \frac{d\varphi}{ds} = \frac{du}{dx}.$$

Other clasp problems. For the simple clasp described above, each component turns a total of $180^\circ$, meaning that $u$ ranges from $-1$, through 0 at the tip, to 1. We can also
Figure 11: In this variant of the simple clasp problem, the endpoints of the two ropes are constrained to lie in four planes whose normals make angle $\arcsin \tau$ with the horizontal. The parameter $u = \sin \varphi$ ranges from $-\tau$ to $\tau$ along each arc, as shown at the end of the top right arc. If extended, the four planes shown would form the sides of a tetrahedron; both arcs are constrained to stay within this tetrahedron.

consider more general clasp problems where the four ends of rope are not vertical (being attached to horizontal planes) but instead are pulled out at some angle (being attached to tilted planes).

Given $0 \leq \tau \leq 1$, we define the $\tau$-clasp to be a problem like the simple clasp where the arc $\gamma$ starts at $u = -\tau$ and then turns through angle $2\arcsin \tau$ to reach $u = \tau$. Our critical $\tau$-clasps have the same 2*2 symmetry as the simple clasp. To put the $\tau$-clasp into our framework of generalized links, we constrain the four endpoints to four planes, each making angle $\arcsin \tau$ with the vertical, as in Figure 11. The complement of the tetrahedron formed by these four planes acts as an obstacle for both curves. The simple clasp is the $\tau$-clasp with $\tau = 1$, where the tetrahedron degenerates to a slab.

**Struts between perpendicular planes.** Whenever two curves in perpendicular planes are connected by a Gehring strut, elementary trigonometry gives us first order information about the curves at both endpoints. We state a general lemma, which we will use here for the clasp and again for the Borromean rings.

Let $P_1$ and $P_2$ be two planes meeting perpendicularly along a line $\ell$, and let $\gamma_i \subset P_i$ be two components of a link. At a point $p_i \in \gamma_i$, we write $x_i$ for the distance from $p_i$ to $\ell$, and $u_i$ for the cosine of the angle between $\ell$ and the line tangent to $\gamma_i$ at $p_i$. These quantities generalize the $x$ and $u$ of Lemma 9.2 above.

**Lemma 9.3.** Let $\gamma_1$ and $\gamma_2$ be two components of a link $L$, lying in perpendicular planes. Suppose there is a strut $\{p_1, p_2\}$ of length 1 connecting these components. Then in the notation of the previous paragraph we have $0 \leq x_1 \leq u_1 \leq 1$, and any two of the numbers $x_1, x_2, u_1, u_2$ determine the other two, according to the formulas

\[
\begin{align*}
x_i^2 &= 1 - \frac{x_j^2}{u_j^2} = \frac{u_i^2(1 - u_j^2)}{1 - u_i^2 u_j^2}, \\
u_i^2 &= 1 - \frac{x_j^2}{u_j^2} = \frac{x_i^2}{1 - x_j^2},
\end{align*}
\]

where $j \neq i$.

**Proof.** Picking cartesian coordinates such that $\ell$ is the $z$-axis and $P_i$ are coordinate planes, we find the strut difference vector $p_1 - p_2$ is $(x_1, x_2, \Delta z)$, for some number $\Delta z$. Since this strut has length 1 and is perpendicular to each $\gamma_i$, we have

\[
\Delta z^2 + x_1^2 + x_2^2 = 1, \quad \Delta z = x_i \frac{u_i}{\sqrt{1 - u_i^2}}.
\]

Simple algebraic manipulations, eliminating $\Delta z$, lead to the equations given.

Note that the condition $x_i \leq u_i$ is exactly the condition that the unit normal circle around $p_i$ intersect $P_j$; the two points of intersection are mirror images (across $P_i$), with the same $x_j$ and $u_j$ values. Also note that we don’t need to have $\gamma_i \subset P_i$ in the lemma; it suffices that $\gamma_i$ be tangent to $P_i$ at $p_i$.

Whenever we have a pair of curves in perpendicular planes, which stay a constant distance 1 apart, we can apply this lemma everywhere along the curves. Each curve $\gamma_i$ is determined as the intersection of the plane $P_i$ with the unit-radius tube around the other curve $\gamma_j$. This will be the situation for the clasp.

**The balancing equations for the clasp.** By Theorem 8.4, in a critical clasp the curvature force along $\gamma$ must be balanced by struts to $\gamma$. In particular, almost every point (indeed, since the set of struts is closed, every point) point $\gamma(u)$ along the curved arc of $\gamma$ must have a strut to some point $\tilde{\gamma}(u^*)$. Then by symmetry we actually have what we call 2-to-2 contact: there are struts from $\gamma(\pm u)$ to $\tilde{\gamma}(\pm u^*)$. Here the two points $\tilde{\gamma}(\pm u^*)$ must be the intersection of the unit normal circle around $\gamma(u)$ with the $yz$-plane, implying that $u^* \in [0, 1]$ is uniquely determined for each $u$. We will refer to $\gamma(u)$ and $\tilde{\gamma}(u^*)$ as conjugate points on the $\tau$-clasp. Lemma 9.3 applies to any pair of conjugate points, with $u_1 = u$, $u_2 = u^*$ and $x_i = x(u_i)$.

**Lemma 9.4.** Suppose $\gamma$ is a planar curve, symmetric across a line $\ell$ in the plane. Consider the net curvature force of a mirror image pair of infinitesimal arcs of $\gamma$. This acts in the direction of the line $\ell$, with magnitude $2|\Delta u|$. Here the function $u$ is defined along $\gamma$ as the cosine of the angle $\psi$ between $\ell$ and the tangent line to $\gamma$.

**Proof.** One infinitesimal arc has net curvature force $k N \, ds = N \, du$. When this is added to the mirror image force, only the component along $\ell$ survives. We get magnitude $2|\sin \psi \, du| = 2|\Delta u|$.

Suppose now we have a symmetric configuration of the clasp where the curved arcs of the two components stay a
constant distance 1 apart. By symmetry we get the 2-to-2 strut pattern described above. Assuming the straight ends of each component meet the constraint planes perpendicularly, our balance criterion Theorem 8.4 says that strong criticality is equivalent to the statement that the net vertical curvature force exerted by the arcs at $γ(±u)$ balances that of the conjugate arcs at $γ(±u^*)$. That is, using Lemma 9.4, for a critical clasp we must have $|du| = |du^*|$, meaning that either $u - u^*$ or $u + u^*$ is constant.

If $u - u^*$ were constant, by symmetry it would be zero, and our equations would describe a pair of half-ellipses, with horizontal major axis $\sqrt{2}$ and vertical minor axis 1. On these curves, corresponding points $γ(u)$ and $γ(u^*)$ are always at distance 1 from each other, but these are maxima for the distance between components, rather than minima. This configuration has $GThi < 1$, and is not $GThi$-critical: the pairs $\{γ(u), γ(u^*)\}$ are not struts.

Instead we must have that $u + u^*$ is constant. To find the constant, note that on the $τ$-clasp, the tip of $γ$ (at $u = 0$) is joined by a strut to the end of $γ$ (at $u^* = τ$); thus $u + u^* = τ$. This equation holds when $0 ≤ u, u^* ≤ τ$; to allow for negative values (parametrizing the whole clasp curve) we write

$$|u| + |u^*| = τ.$$  

We can now give an explicit description of our critical $τ$-clasp:

**Theorem 9.5.** Let $τ ∈ [0, 1]$, and let $γ = γ_τ$ be the curve in the $xz$-plane given parametrically for $u ∈ [−τ, τ]$ by

$$x = x_τ(u) := \frac{u\sqrt{1 − (τ − |u|)^2}}{\sqrt{1 − u^2(τ − |u|)^2}},$$

$$z = z_τ(u) := \int \frac{dτ}{d x} \, dx = \int \frac{u}{\sqrt{1 − u^2}} \kappa_τ(u),$$

where

$$\kappa_τ(u) := \frac{\sqrt{(1 − u^2(τ − |u|)^2)^3(1 − (τ − |u|)^2)}}{1 − (τ − |u|)^2 + (τ − |u|)|u|(1 − u^2)}$$

and the constant of integration for $z$ is chosen so that

$$z(0) + z(τ) = −√1 − τ^2.$$  

Then the union of $γ$ with its image $γ_τ$ under the symmetry group 2*2 described above is a $τ$-clasp that is strongly critical for Gehring ropelength. The curvature of $γ$ is $κ(u)$ above, and the total length of the curved part of $γ$ is

$$\int_{−τ}^{τ} \frac{du}{\kappa_τ(u)} \sqrt{1 − u^2}.$$  

**Proof.** The proposition follows from the foregoing discussion, after substituting $u^* = τ − |u|$ into the equations of Lemma 9.3, and using Lemma 9.2. To get the constant of integration for $z$, we note that the strut from $γ(0)$ to $γ(τ)$ has height given (as in the proof of Lemma 9.3) by

$$Δz = \sqrt{1 − x_τ(0)^2 − x_τ(τ)^2} = \sqrt{1 − 0 − τ^2}.$$  

\[\square\]

Figure 12: This is an accurate plot of the Gehring-critical simple clasp $γ$ given by Theorem 9.5. Here $u = \sin φ$ ranges from $−1$ to $1$ over the curved portion of $γ$. The tip $γ(0)$ of the other component is shown above $γ$ on the $z$-axis, along with the (dotted) circular cross-section of the tube of unit diameter around $γ$. The curved lines extending down from the sides of this cross-section are the lines of contact between the shaded tube around $γ$ and the front half of the tube around $γ$. Symmetric lines of contact extend behind the shaded tube, realizing the 2-to-2 contact pattern we have described. Finally, we see a small gap between the tubes, explored in more detail in Figure 14.

Although the formulas we have given for $z_τ(u)$ and for arc-length both involve hyperelliptic integrals not expressible in closed form, it is straightforward to integrate them numerically; we have plotted our critical configuration of the simple ($τ = 1$) clasp in Figure 12.

As we mentioned in the Introduction, Starostin has announced [Sta03] an independent derivation—using a notion of balancing for smooth curves—of these same $τ$-clasp configurations (as well as the stiff clasps we will consider in [CFK*04]). Starostin does not prove that these configurations are critical for Gehring ropelength.

**The geometry of the Gehring clasp.** We now examine the curvature and other geometric features of the critical clasps given in Theorem 9.5. Each component of the critical $τ$-clasp is a $C^1$ join of four analytic pieces: a straight segment, then $γ[−τ, 0]$, then $γ[0, τ]$, and finally another straight segment. Where the curved arcs join the straight segments at $u = ±τ$, the curvature $κ(u)$ approaches 1; at these points, our critical clasp agrees to second order with the naively expected circular arcs.

The maximum curvature $κ(0) = 1/\sqrt{1 − τ^2}$ occurs at the tip. For $τ < 1$, this is finite, and our $τ$-clasp is $C^{1,1}$. But for $τ = 1$, the curvature blows up (like $|s|^{−1/3}$) at the tip. In Figure 13 we plot the curvature $κ(u)$ for our Gehring simple clasp. This curve is $C^{1,2/3}$ (and is also in the Sobolev space $W^{2,3−ε}$ for all $ε > 0$) but has no higher regularity.

In Proposition 6.5, we proved that Gehring-critical curves are $C^3$. It would be interesting to find out whether all Gehring-critical curves are $C^{1,2/3}$; perhaps the simple clasp exhibits the worst possible behavior.
In Example 7.4, we saw Gehring-critical curves which fail to have positive thickness in the ordinary sense of [CKS02] because one component is nonembedded. The Gehring simple clasp fails to have positive thickness for a different reason: its curvature is unbounded. In [CFK+04] we will consider a family of thickness measures with a variable stiffness parameter $\lambda$. In these measures, a unit-thickness curve has curvature bounded below $2/\lambda$. For any nonzero $\lambda$, the critical simple clasp must be different from the Gehring clasp, and must instead include an arc of the maximum allowed curvature.

One of the most interesting features of the clasp is the gap between the two components of the clasp. The distance between the tips of $\gamma$ and $\overline{\gamma}$ is $z(\tau) - z(0) + \sqrt{1 - \tau^2}$ (written in this way to be independent of the constant of integration for $z$). This is an increasing function of $\tau$, close to 1 when $\tau$ is small, but increasing to 1.05639 at $\tau = 1$. Thus, in the Gehring simple clasp, the gap between the thick tubes around the two components, at their tips, is almost 6% of their diameter.

These thick tubes contact each other at the midpoints of the Gehring struts. The lines of contact form a loop with four vertical cusps, shown in Figures 12 and 14. The solid tubes divide the rest of the ambient space into two regions: one infinite component around the outside of the clasp, and one small chamber sitting in the gap between the tips, shown in Figure 14. To give a sense of scale, the gap chamber has a substantial surface area of about 1.10, equal to the area of a section of tube of length more than 1/3. However, the chamber is very thin, resulting in a volume of only 0.01425.

**Length comparison with the naive clasp.** Earlier, we described the naive circular configuration for the simple clasp. Similarly, in what we call the *naive $\tau$-clasp*, each component is built from straight segments (normal to the constraint planes) and a unit-radius arc (of angle $2 \arcsin \tau$ and centered at the tip of the other component). As we saw for $\tau = 1$, this configuration is not Gehring-critical: there is no way to balance the forces concentrated on the tips, unlike in Examples 7.1 and 9.1, which had extra struts.

Our critical $\tau$-clasps (which we expect are the global minima for length) are indeed slightly shorter than the naive configurations. The total length of a clasp depends, of course, on the position of the bounding planes. Thus to compare the lengths of the naive clasp and our critical clasp in a meaningful way, we introduce the notion of excess length. The infimal possible length of a $\tau$-clasp with no thickness constraint is easily seen to be four times the inradius of the bounding tetrahedron. (In the case $\tau = 1$ this is twice the thickness of the bounding slab.) The *excess length* of any given clasp is the amount by which its length exceeds this value.

For $\tau = 1$, the naive clasp has excess length $2\pi - 2$, since two unit semicircles replace two straight segments of unit length. Numerical integration reveals the excess length of our critical 1-clasp to be 4.262897 (accurate to the number of digits shown). It is thus about 0.020288, or half a percent, shorter. In general, the excess length of the naive $\tau$-clasp is $4 \arcsin \tau - 2\tau$, while the excess length of our critical $\tau$-clasp equals the total length of the curved parts minus $2\tau$ times the inter-tip distance. The maximum percentage savings, about 0.518%, occurs for $\tau \approx \sin(80^\circ)$.

10. **THE BORROMEO RINGS**

The original Gehring link problem was solved by the Hopf link made from a pair of circles through each other’s centers. We have already generalized this to a three component link in one way: the simple chain made from circles and stadium curves of Section 7. But the simple chain is just a connect sum of Hopf links, and so the minimizing configuration shares much of its geometry with the original Gehring solution.

We now construct a proposed minimizer for a more interesting Gehring problem—the Borromean rings. Among the three prime six-crossing links of three components, the Borromean rings (shown in Figure 15) is the one which is Brunnian, meaning that if any one component is removed the remaining components are unlinked. Milnor’s $\mu$-invariant classifies three-component Brunnian link-homotopy types, and the Borromean rings are the first nontrivial example.

In this section, we describe (Theorem 10.2) a critical configuration $B_0$ of the Borromean rings. Numerical simulations with Brakke’s *Evolver* [Bra92] suggest that this configuration $B_0$ is in fact the ropelength-minimizing Borromean rings. We will see below that the curvature of $B_0$ stays below 1.534; this means (as we show in [CFK+04]) that $B_0$ is also a critical point for length when constrained by the ordinary thickness measure of [CKS02] instead of by Gehring thickness. In [CKS02], we described a similar configuration $B_2$ of the Borromean rings, built entirely from arcs of unit circles. Theorem 6.1 shows that $B_2$ is not Gehring-critical, and we compute that $B_0$ is 0.08% shorter.

**Symmetry and convexity.** Our configurations $B_0$ and $B_2$ of the Borromean rings are quite similar, and in particular have the same symmetry and convexity properties, which we now...
not at the same distance from the origin. We will assume that \(\tau\) is the distance between the tubes at the tips of the clasp. The grid in the center and right pictures is a square grid projected from the \(xy\)-plane. On the right, we see a tiny ridge running from left to right along the surface of the chamber; this is a cusp formed by the folding of the tube surface that happens when the curvature of the clasp rises above 2 (cf. Figure 13). We do not know whether this gap chamber forms in clasps of physical rope; it would be very interesting to find out.

The configuration built from circular arcs. The configuration \(B_2\) we described in [CKS02] is generated by an arc \(IT\) of this form. In \(B_2\), we have \(R = T\), so that the entire convex arc \(JT\) is part of the unit circle around \(\hat{I}\). Furthermore, the concave arc \(JI\) is also part of a unit circle, centered at \(\hat{T}\). This implies that \(M = J\) and \(\sigma = \rho =: \rho_2\), determined by the fact that \(I\) and \(\hat{T}\) are at unit distance, meaning...
ordinates of $M$ is concave and symmetry is generated by a planar arc $IT$. There are points $M$ and $\tilde{M}$, and $I$ and $\tilde{I}$ are the rotation images of $M$ and $\tilde{M}$, respectively. Any configuration $B$ of the Borromean rings with $3\times 2$ symmetry is generated by a planar arc $IT$. We consider arcs where $IJ$ is concave and $JT$ is convex. The other points of $B$ in this quadrant are the rotation images $I$ and $\tilde{T}$ of $I$ and $T$. In our configurations, there are points $M$ and $R$ such that $JR$ is part of the unit circle around $J$, and $M$ is the midpoint between $I$ and $\tilde{I}$. The four dotted lines are thus struts of length 1. The height difference from $J$ to $I$ is $\sigma = \sin \psi(J)$ as delineated by the horizontal dashed line, and the coordinates of $M$ are given in terms of $\rho = \sin \psi(M) = -\cos \varphi(M)$.

$$2\rho_2 + 1 = 2\sqrt{1 - \rho^2}.$$ As we computed in [CKS02], the total length of $B_2$ is then $6\pi + 24 \arcsin \rho_2 \approx 29.0263$.

This configuration is not balanced (and thus not critical) for Gehring ropelength. To balance the curvature forces of the circular arcs, the fans of struts to their centers would have to carry force proportional to arclength. But these struts would then concentrate outward force on the tips and inward force on the intips; there are no further struts to balance these forces. This is like the picture for the naive clasp—all the force is concentrated on the tips. As for the clasp, the tips in the Gehring-critical configuration will be further apart.

In [CFK+04], we introduce a family of thickness measures with variable stiffness. For stiffness 2 (meaning that the curves cannot have osculating circles of diameter less than 2) we will see that $B_2$ is balanced and hence critical for rope-length. Because the circular arcs have exactly the maximum allowed curvature, we will see that their curvature force need not be balanced pointwise, but only in total. Outward strut force on their midpoints (the tips and intips) can in a sense be spread out to balance the curvature all along the arc. Because $\rho_2 \neq 45^\circ$, however, there is an imbalance of total curvature forces between the convex and concave arcs. Thus our balancing measure will need an atom of force on the special colinear struts $\{I, M\}$ and $\{M, \tilde{T}\}$; this transmits force from $\tilde{T}$ through $M$ to $I$.

**The critical configuration.** To get a balanced configuration $B_0$ of the Borromean rings, we have to replace the concave circular arc $IJ$ (and part of the convex arc) by a Gehring clasp arc. Suppose $IJ$ is part of a $\tau$-clasp for some $\tau \geq \sigma$. We will now describe a configuration determined by certain values of our three parameters

$$0 \leq \rho \leq \sigma \leq \tau \leq 1,$$

a particular curve of the class illustrated in Figure 16.

First, the arc $IJ$ is the piece $v \in [0, \sigma]$ of the $\tau$-clasp, translated out along the $x$-axis until its tip $I$ is at $(2\rho, 0, 0)$. Next, $JMR$ is an arc of the unit circle around $\tilde{I}$, with $\psi(J) = \sigma$, $\psi(M) = \rho$ and $u(R) = \tau$. Note that to get these arcs to match up at $J$, we will need two conditions on our parameters $\rho$, $\sigma$ and $\tau$. Finally to define the remaining arc $RST$, consider the image $IJM$ of $IJM$, rotated into the $yz$-plane. Then $RST$ is conjugate to $IJM$ in the sense of Lemma 9.3: it is the intersection of the unit-radius tube around $IJM$ with the $xy$-plane, with $S$ defined to be the point conjugate to $J$. Figure 17 shows the arc $IT$ and its two rotated images, that is, the part of $B_0$ lying in the positive orthonth in space.

**Lemma 10.1.** Suppose the parameters $0 \leq \rho \leq \sigma \leq \tau \leq 1$ satisfy the two equations

$$0 = 2\rho - \sqrt{1 - \sigma^2} + \int_0^{u = \sigma} \frac{u \, du}{\kappa_r(u) \sqrt{1 - u^2}} \quad (10.1)$$

and

$$0 = 1 - (2\rho - \sigma)^2 - \frac{1 - \sigma^2}{1 - \sigma^2 (\tau - \sigma)^2} \quad (10.2)$$

where $\kappa_r$ is the curvature of the clasp from Theorem 9.5. Then there is a $C^1$ and piecewise analytic arc $IJMRST$ as described in the last paragraph. Its images under the symmetry group $3\times 2$ form a configuration $B(\rho, \sigma, \tau)$ of the Borromean rings with Gehring thickness $GT_{\text{Thi}} = 1$.

**Proof.** As a point on the unit circle $JR$ around $\tilde{I}$, the jump point $J$ has coordinates

$$(\sqrt{1 - \sigma^2}, 2\rho - \sigma, 0).$$

As a point on the $\tau$-clasp $IJ$, its coordinates are

$$(2\rho + \int_0^\sigma \frac{u \, du}{\kappa_r(u) \sqrt{1 - u^2}}, x_r(\sigma), 0).$$

Equating these, using

$$x_r^2(\sigma) = 1 - \frac{(1 - \sigma^2)}{1 - \sigma^2 (\tau - \sigma)^2}$$

from Theorem 9.5, gives (10.1) and (10.2).

If these equations are satisfied, then the position of $J$ is well-defined, and $IJR$ is a $C^1$ arc, meeting the $x$-axis perpendicularly. The arc $RST$ is the conjugate of $IJM$ and thus is $C^1$ by Lemma 9.3. At $T$, the same lemma shows it meets the $y$-axis perpendicularly. At $R$, the $u = \tau$ base of the $\tau$-clasp agrees even to second order with the unit circle.
In this configuration, all the struts shown in Figure 17 have length 1. If the Gehring thickness were less, there would need to be some shorter strut in this positive octant. But that strut would be governed by Lemma 9.3, and its projection to the $xy$-plane would be normal to the arc $IT$; the figure makes it clear that no such strut exists.

The balance criterion. Finally, we wish to find the third condition on our parameters $\rho$, $\sigma$ and $\tau$, which will ensure that $B(\rho, \sigma, \tau)$ satisfies the balance criterion of Corollary 6.3.

For most of the struts, it is immediately clear what stress they need to have in a balancing measure $\mu$: The struts from $IJ$ to $\tilde{R}\hat{S}$, and those from $MR$ to $\tilde{I}$ and from $RT$ to $\tilde{I}M$ must be stressed exactly enough to balance the curvature force of $IJ$ and of $MRT$. The conjugate clasp arcs $IJ$ and $RS$ exactly balance each other’s curvature forces in this way.

The situation along the short circular arc $JM$ is more complicated. The struts inward to $\tilde{I}$ need to balance not only the curvature force of $JM$ itself, but also the force acting inwards on $JM$ from the struts from $\tilde{ST}$. Remember that the measure needed on these last struts is determined by the curvature of $\tilde{ST}$; this in turn determines the measure needed on the inward struts from $JM$. We will write this down explicitly below. The final condition on our parameters then comes from a balance of forces at $\tilde{I}$, where a whole family of struts converge.

Note that these Borromean rings $B_0$ form the first known example of a ropelength-critical configuration in which this sort of transmitted force appears. Struts impinge on the arc $JM$ from the direction opposite its own curvature, and transmit their force through that arc. Without this force transmitted through the (very) short arc $JM$, the relatively long convex piece $\tilde{RT}$ would exert too much inwards force on the relatively short concave piece $IJ$. Instead, some of this inwards force, when transmitted through $JM$, becomes force outwards on the concave piece $IJ$. This transmitted force plays the same role in balancing $B_0$ that the atomic force from $\tilde{T}$ through $M$ to $\tilde{I}$ played in balancing $B_2$ for the stiff problem. But here our strut measure is absolutely continuous, with no atoms.

To write down the final balancing condition at $\tilde{I}$, we begin with an application of Lemma 9.4: the total curvature force of $JMRT$ and its mirror image across the $yz$-plane acts on $\tilde{I}$ downward in the $y$-direction, with magnitude

$$2(u(J) - u(R)) = 2(\sqrt{1 - \sigma^2} - \tau).$$

But the struts from $JM$ carry extra transmitted force. To determine this, consider the curvature force of infinitesimal arcs of $\tilde{ST}$ and its mirror image across the $xy$-plane. Parametrizing them as usual by $u$, Lemma 9.4 tells us the net force, exerted in the negative $x$ direction, is $2du$. This horizontal force is exerted on an infinitesimal piece of $JM$ and its mirror image across the $xz$-plane. If we parametrize $JM$ by $v = \sin \psi$, then remembering that the force on this arc acts perpendicular to the arc, we see that if its horizontal component is $du$, then its vertical component is $v du/\sqrt{1 - v^2}$. This force gets transmitted through to $\tilde{I}$. Because of the symme-
try across the $yz$-plane, of course only the vertical component matters in the end. But this symmetry also doubles that vertical force. (Four copies of the arc $ST$ act on $I$: the original, and reflections across the $xy$- and $yz$-planes.) The resultant total transmitted force on $I$ is upwards with magnitude

$$2 \int_{u=0}^{\tau-\sigma} \frac{2v}{\sqrt{1-v^2}} \, dv.$$ 

Here the upper limit of integration is $u(S) = \tau - v(J)$ because $J$ and $S$ are conjugate points on the $\tau$-clasp. To make this integral explicit, we need to give the relation between $u$ and $v$; this comes from Lemma 9.3. Along $JM$ we have $y$-coordinate $2\rho - v$, so the lemma gives

$$u^2 = u(v)^2 := \frac{1 - (2\rho - v)^2/v^2}{1 - (2\rho - v)^2}.$$ 

If one wanted, this could be solved to give $v$ as the root of a quartic equation in $\rho$ and $u$. Note that $u = 0$ at $v = \rho$, as we expect for $T$ and $M$. Plugging in $u = \tau - \sigma$ and $v = \sigma$ (at $S$ and $J$) reproduces (10.2).

Summarizing, we can write the force-balancing condition at $I$ as

$$0 = \tau - \sqrt{1 - \sigma^2} + \int_{\rho = \tau}^{\sigma} \frac{2v}{\sqrt{1-v^2}} \, dv (u(v)) \, dv \quad (10.3)$$

and so we have proved

**Theorem 10.2.** Suppose $\rho = \rho_0$, $\sigma = \sigma_0$ and $\tau = \tau_0$ satisfy the three equations (10.1), (10.2) and (10.3). Then the Borromean rings $B_0 = B(\rho_0, \sigma_0, \tau_0)$, constructed as in Lemma 10.1, are strongly critical for Gehring ropelength. □

It is easy to solve (10.1) for $\rho$, or (10.2) for $\rho$ or $\tau$, or (10.3) for $\tau$, thereby eliminating one of our three variables. Then we are left with two nonlinear integral equations in the other two variables. While we have not proved formally that a solution to this system exists, we have solved it numerically to high precision, both in Mathematica and using QUADPACK/MINPACK [PdDUK, MGH]. We obtain

$$\rho_0 = 0.4074218, \quad \sigma_0 = 0.4177486, \quad \tau_0 = 0.7561107,$$

where again we follow the standard convention that the error is less than $\pm 1$ in the last digit shown. There is nothing delicate about this solution, since our expressions vanish to first order at this point. Numerically it is also clear that this solution is unique.

Using these constants, we compute the length of our critical Borromean rings $B_0$ as 29.0030. By comparison, the length of the piecewise circular Borromean rings $B_2$ was 29.0263. Thus our critical configuration $B_0$ beats the naive circular configuration $B_2$ by slightly less than one-tenth of one percent. For comparison, the best known lower bound for the length of the Borromean rings is $6\pi$ [CKS02].

In Figure 18 we plot the curvature of the critical Borromean rings $B_0$ as a function of arclength. Note that it is discontinuous only at $J$ and $S$. Each component in $B_0$ is built of 14 analytic pieces, joined in a $C^{1,1}$ fashion at the symmetric images of the points $I$, $J$, $R$, and $S$. The maximum curvature (at the intip $I$) is $(1 - \tau_0^2)^{-1/2} \simeq 1.528$. Therefore $B$ is also ropelength critical for the standard ropelength functional of [CKS02], as we will show in [CFK+04]. It is also critical for all the stiff ropelength functionals where the lower bound $\lambda$ on the diameter of curvature is less than $2\sqrt{1 - \tau_0^2} \simeq 1.3$.

We note that Starostin has described [Sta03] a configuration $B_S$ of the Borromean rings with ropelength intermediate between that of our $B_2$ and $B_0$; his configuration replaces the arcs $IJ$ and $RT$ of $B_2$ by clasp arcs, but does not incorporate the other features of $B_0$. While $B_S$ can be balanced almost everywhere and Starostin appears to assume that it is a critical configuration, in fact it is not balanced at the intips since it does not satisfy the equivalent of (10.3). Thus by Corollary 6.3, $B_S$ is not critical.

11. OPEN PROBLEMS AND FURTHER DIRECTIONS

Our work in this paper has been motivated by a simple principle: extending the ideas of rigidity theory for finite frameworks of bars and struts to handle mechanisms built from continuous curves of constraints and contacts. In the simple case of Gehring-ropelength critical links, this method has already yielded some powerful results, such as our $C^1$ regularity theorem, and some real surprises, like the Gehring clasp and the critical Borromean rings. Furthermore we expect that these methods in general, and our Kuhn–Tucker Theorem 5.4 in particular, will prove to be a useful tool, with applications to a number of outstanding problems in the geometry and topology of curves and surfaces.

We have mentioned our forthcoming extension of these re-
sults [CFK +04] to the classical ropelength problem, where the presence of curvature constraints and self-contacts of the tube around individual components makes the situation considerably more challenging. Our theory of generalized links and obstacles should also be applicable to the study of packing problems for tubes and surfaces, as when thick rope is packed into a box (a problem of some interest in molecular biology: see [MBMS99, MMTB00]), or when the gray matter of the brain is folded and pressed against the skull. We should also mention that while here we have only considered minimizing length in this work, our framework should work equally well for other objective functionals, such as a general theory of elastic rods with self-contact.

And we must mention the promise of Connelly, Demaine, and Rote’s fascinating Unfolding Theorem [CDR03].

**Theorem 11.1.** Any embedded, non-convex planar polygon admits a motion that preserves all edge lengths and strictly increases the distance between any two points on the polygon not already joined by a straight line of polygon edges.

One key step in their proof is based on a finite-dimensional duality theorem akin to our Kuhn-Tucker theorem. So our theory allows us to complete part of the proof of the (conjectured) generalization to smooth plane curves. Whether our methods can be made strong enough to overcome the formidable difficulties involved in proving a smooth unfolding theorem remains to be seen.

There are several specific open questions suggested by our work above. Most importantly,

**Question.** What is the regularity of a Gehring-critical curve? Our work above shows that such curves are at worst $C^1$ and at best $C^{1,2,3}$.

While we have demonstrated critical configurations of the Gehring clasps and Borromean rings, we have not attempted to prove that these configurations are minimal.

**Question.** Are our Gehring clasps and Borromean rings length-minimal in their link-homotopy types?

The Euclidean-cone methods of [CKS02] seem to hold out some hope for reducing the clasp problem to the case where both curves are planar, but we have not investigated this line of attack.

**Acknowledgments**

We gratefully acknowledge helpful conversations with many colleagues, including Ted Ashton, Bob Connelly, Erik Demaine, Elizabeth Denne, Oğuz Durumeric, Oscar Gonzalez, Xiao-Song Lin, Marvin Ortel, and Heiko von der Mosel. In addition, we would like to thank John Maddocks and the Bernoulli Center at EPFL for hosting some of us during some of the summer of 2003, Eric Rawdon and Michael Piatek for sharing their TOROS-minimized link data, and Matt Hoffman for helpful technical advice on the (extraordinary) Electric Image rendering system. Some of the other figures in the paper were prepared with Geomview. This work has been partially supported by the NSF through grants DMS-99-02397 (to Cantarella), DMS-02-04826 (to Cantarella and Fu), DMS-00-76085 (to Kusner) and DMS-00-71520 (to Sullivan).


[CS99] Cristian Micheletti, Jayanth Banavar, Amos Maritan, and F. Seno, Protein structures and optimal folding from a geometric variational principle, Physical Review Letters 82 (1999),
3372–3375.


