# VISUALIZATION OF SOAP BUBBLE GEOMETRIES

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Received 29 June 1990

ABSTRACT

The authors discuss mathematical soap bubble problems and a new technique for generating computer graphics of bubble clusters. The rendering program is based on Fresnel's equations and produces both the colored interference patterns of reflected light and the Fresnel effect of varying transparency.

A single soap bubble possesses an exquisite perfection of form. Soap bubbles are lovely physical manifestations of simple geometric relationships created by the principles of area minimization. Our goals in studying soap bubble problems are both to better understand problems of area minimization and to use such problems as test cases for the computation and visualization of geometric structures which arise in other optimization problems. In this article we report in particular on new techniques for displaying soap bubble geometries; these techniques incorporate both colored interference patterns and the Fresnel effect of decreased transparency at oblique angles.

## Minimal surface forms.

A collection of surfaces, interfaces, or membranes is called a 'minimal surface form' when it has assumed a geometric configuration of least area among those into which it can readily deform. Of course there must be some constraints in the problem to keep the configuration from collapsing completely. Typical constraints might be a fixed boundary wire the surface must span, or a volume it must enclose. The sphere, the shape of a single soap bubble, seems the simplest minimal surface form; it has least area among all surfaces which enclose the same volume.

Minimal surface forms arise not only in the surface tension phenomena of liquids and thin films, such as soap bubbles, but also in grain boundaries in metals, in radiolarian skeletons, in close packing problems, in immiscible liquids in equilibrium, in sorting of embryonic tissues, in design, in art, and in mathematics [1].

## Mathematical principles of soap bubble geometry.

Soap bubble clusters consist of regions of trapped air separated from each other and from the outside by smoothly curved surfaces. Surface tension tends to minimize surface area, pulling each of these surfaces tight. This tendency of surface tension to collapse the cluster is held in balance by the differing pressures of the trapped regions of air and the outside atmosphere. A soap bubble in the shape of a sphere thus has higher-than-atmospheric pressure on its inside. If a soap bubble interface curves in two different directions (like the seat of a saddle), then the higher pressure lies on the side of the greater curvature and the difference in pressure is proportional to the difference between the two curvatures. The fact that pressures are constant within each region means that the average, or mean, curvature of the interface surface between any two regions must be the same at each point of this interface: the net compressional force generated by such an interface is proportional to the mean curvature, and it must exactly balance the pressure difference.

As an example, if in $(x, y, z)$ space an interface in a soap bubble cluster passes through the origin $x = y = z = 0$ and can be written nearby as the graph of the equation

$$z = f(x, y) = 9x^2 - 5y^2 + \text{ higher-order terms in } x \text{ and } y$$

for small values of $x$ and $y$ (so that it is saddle-shaped, curving down in the $y$ direction but more sharply up in the $x$ direction), then it would have mathematical mean curvature equal to $9 - 5 = 4$, and the air pressure above the interface (*i.e.,* for $z > f(x, y)$) would exceed the pressure immediately below by an amount proportional to this mean curvature. One of the central difficulties in the mathematical analysis of phenomena like soap films is that the functions which describe them (such as $f(x, y)$ above) usually cannot be written down completely in any finite way. Thus theorems about such geometries say that such functions exist and have certain properties, but they rarely are able actually to exhibit the functions.

Notwithstanding the difficulties of writing down soap bubble configurations exactly, mathematical analysis has shown a great deal about the possible structure of minimal surface forms. It guarantees that any such form must consist of a finite number of smoothly curving sheets of surface each with constant mean curvature. These are allowed to meet only along a finite number of smoothly curving arcs, where exactly three sheets must meet at equal angles of 120°. Finally, these arcs can meet only at a finite number of points, where four of these arcs come together, always in the same pattern. The three surfaces along each arc make a total of six surfaces meeting at the point. Since the sheets must meet at 120° angles, the arcs necessarily meet at approximately 109° angles—the dihedral angles of a regular tetrahedron. These principles were made mathematically rigorous by J. E. Taylor [2].

## The computation of soap bubble geometries.

The problem of computing soap bubble geometries is a difficult one which has not yet been completely solved [3]. Because the surfaces usually cannot be described exactly it is necessary to approximate them. One way to generate an approximate minimal surface is to make a guess as to what the configuration should be and write a simple combinatorial description of it as a triangulated surface, specifying which edges are to be fixed as boundary wires and which volumes should be maintained. Then, keeping the same combinatorial configuration, the vertices can be moved to decrease area.

A convenient computer program for doing this, the Surface Evolver, has been written at the Geometry Supercomputer Project [4]. The program is interactive—it can alternate between moving the vertices to minimize area and refining the triangulation to enable the surface made up of triangles to more closely approximate the true smooth surface.

Geometries generated by this evolver program should approximate smoothly curved mathematical minimal surfaces. But no test is known which could ensure this or determine when the computed geometries correspond to physical soap bubble clusters. The latter is especially tricky because of the difficulty of blowing even relatively simple real soap bubble clusters.

**Rendering soap bubble geometries.**

An accurate computer approximation to a soap bubble configuration involves a list of the positions of several thousand vertices and the combinatorial relationships between the associated edges, triangles and solid regions. To make any sense of this information, it is necessary to translate it into pictures. (In an interactive program like the evolver, graphics are used extensively before the final data file is obtained.) Since there are a number of different surfaces in even simple soap bubble clusters, transparency is necessary in the rendering, in order to see the inner interfaces. Motion can be useful in visualizing any three-dimensional object: it helps the eye pick out the spatial relationships between different surfaces. The present article, of course, cannot illustrate the effect of motion on visual understanding [5].

Despite the importance of motion to the eye, there are ways to make even a static image more effective. One possible technique is the use of stereoscopic pictures [6]. For certain pictures (renderings of complex molecules, for example), the effects can be dramatic, and sometimes indispensable. Human eyes are very good at seeing in three dimensions even without stereoscopic information, however.

When one views a real three-dimensional scene, even though the image present on the retina is just two-dimensional, the brain immediately reconstructs a three-dimensional mental picture. (Accurate perspective drawing is difficult because it is hard for the mind to recreate the two-dimensional image.) The cues to the eyes which make this possible are many and complex. Certainly, reflected highlights and textures on surfaces are important. A good artist can pick out these and other details and, with just a few strokes, produce a picture that gives the eyes the necessary clues, while leaving out other details which would not help. Computers cannot yet do this difficult task, but often, if a computer image can be made more realistic, it will provide more subconscious clues to the eye, and the third dimension will be more apparent.

**A soap film rendering algorithm based on Fresnel's equations.**

Simply using "flat transparency" to make the films in a cluster 80% transparent, say, makes inner interfaces visible, but does not make them look like real soap bubbles. One noticeable feature of the appearance of a soap film is the Fresnel effect, which makes the film—like a piece of glass—less transparent when viewed edge-on. Reproduction of this effect [7] conveys the roundness of soap films much better than flat transparency does. In 1990, co-author John Sullivan utilized some fundamental equations of optics to model other features of soap film, creating what seem to us to be striking visual effects.

Most surfaces rendered in computer graphics are assumed to respond to ambient or diffuse light present in the scene. On the other hand, real soap films create nearly mirror-like reflections of objects around them, without any diffuse component. Highlights of windows or bright lights thus appear in curved patterns, which allow the shape of the surface to be seen. Also, optical interference between light rays reflected from the inside and outside of thin soap films creates brightly colored stripes within highlights [8]. The computer images created by our special shading routine show both the highlights and colored bands of real soap bubbles [9].

Figure 1 shows a small cluster of six bubbles rendered by our technique. This cluster is interesting because, although clusters of up to five soap bubbles can be constructed with all the interfaces being pieces of spheres, this one cannot. The squarish interface between the two small bubbles in the center is evidently saddle-shaped, with Gaussian curvature less than zero, and is certainly not part of a sphere. We generated this geometry with the evolver program mentioned before.

For our renderings of computer generated soap films, we have used the RenderMan software from Pixar; it decides which interfaces are in front of which others and combines transparency information. We need merely to write code which, for any small patch on any surface, will compute the transparency of different colors and the amount of reflected light to be added.

We have created an artificial environment that includes several bright windows on the walls and a pattern of lights on the ceiling to provide highlights in the soap film. The colors of soap film are caused mainly by the varying thickness of the film, so real soap bubbles have mainly horizontal color stripes, since gravity pulls the film towards the bottom of the bubble, making it thicker there. But these stripes develop in a complicated way, and have many seemingly random perturbations. Since we know of no good physical model for expecting particular thicknesses at particular points of the film, we have specified the thickness using computer-generated random noise, weighted to give somewhat horizontal patterns. Knowing this thickness, together with the direction of the surface and the direction from which the computer's eye is looking at it, enables us to use physical laws to determine the amounts of transparency and reflected light. The basic laws of electromagnetic waves lead to Fresnel's equations in optics, which tell the fraction of light reflected at an interface. One of the things these equations describe at any boundary between different transparent materials is the Fresnel effect of varying transparenc we have already mentioned—this makes the surface of a pond, for instance, more reflective (and less transparent) at a shallow angle. In addition, when applied to a thin film with two nearby interfaces, the equations give rise to the colored interference patterns seen in oil slicks as well as in soap bubbles. The thickness is the main determinant of color, while the angle of view mainly determines overall opacity or reflectivity. These effects, however, are not really independent—they both arise from the same equations.

In more detail, if $\beta$ is the coefficient of refraction (about 4/3 for a water-air interface), then Snell's law says that the angles of the incident and transmitted (or refracted) rays are related by $\beta = \sin\theta_i / \sin\theta_t$. If we define $\alpha = \cos\theta_t / \cos\theta_i > 1$, then Fresnel's equations say that the intensity of the electric field for the wave reflected off the interface is $r$ times that of the incident wave, where $r = (1-\alpha\beta)/(1+\alpha\beta)$ for normal polarization, and $r = (\beta-\alpha)/(\alpha+\beta)$ for parallel polarization. By the principle of energy conservation, the transmitted field satisfies $t^2 = 1 - r^2$.

In our case, we have a second interface quite close to the first: a soap film is a narrow sheet of water with air on both sides. Transmitted light refracts back to its original direction, so we don't have to worry about bending light, though we calculate $\theta_t$ to compute the reflection coefficients. Light may bounce back and forth any number of times within the soap film, before either being transmitted or reflected. Rays of light with different numbers of internal reflections have a phase shift, proportional to the thickness (in the direction of the internal refracted rays) expressed in numbers of wavelengths. If this phase delay is $\phi$ for a single round-trip, then the strength of the overall transmitted electric field is obtained by summing the geometric series

$$t^2(1 + r^2 e^{i\phi} + r^4 e^{2i\phi} + \cdots).$$

The intensity of transmitted light is the square norm of this sum,

$$T = 1 - \frac{t^4}{t^4 + 2(1-t^2)(1-\cos\phi)}.$$

We have used these equations of physics, derived from those of Fresnel, in our computer program to compute which colors are reflected and which are transmitted. For each piece of surface rendered, RenderMan invokes our program, giving it the incident and reflected directions. We use these to compute the transmission coefficient for both normal and parallel polarizations and for two different wavelengths of each color (red, green, blue). The average coefficient for each color gives the fraction of light transmitted, and goes directly back to RenderMan as the transparency. The reflection coefficient $R = 1 - T$ is multiplied by the intensity of that color of light from whatever window or other feature of our environment is present in the reflected direction, to give the reflected light.

Computer graphics usually uses colors expressed in terms of red, green and blue. We do our calculations for two shades of each, which approximates more closely the full spectrum present in white light, to get accurate renditions of the rainbow of colors in real soap film.

Note that the algorithm used is not a ray tracer so that, in particular, there are no reflections of reflections. The incoming light in the reflected direction is always assumed to be only that from the wall or ceiling in the environment.

**A cluster of 120 regions.**

The Color Plate shows a cluster of 119 soap bubbles (plus one infinite region) rendered by our new technique. This highly symmetrical figure is a stereographic projection of a four-dimensional regular polyhedron. Its structure is perhaps most easily described by analogy in three dimensions.

The analogous three-dimensional polyhedron is the regular dodecahedron illustrated in Fig. 2a—it has 12 flat faces, each of which is a regular pentagon; the pentagons share 30 edges and 20 vertices. Figure 2b shows this dodecahedron pushed out to lie on a sphere. Here the edges meet at the 120° angles of a bubble cluster. In Fig. 2c, the spherical dodecahedron has been stereographically projected onto the plane. Since stereographic projection preserves circles and the angles at which they meet, this picture has the right geometry for a two-dimensional soap bubble cluster in the plane. If we lived in this plane, we would of course have to use transparency to see the many layers in this picture.

In our three-dimensional space we could make models of such planar clusters by blowing bubbles between two nearby flat sheets of glass. The films will extend between the two sheets, meeting them at right angles, and could form a pattern like that in Fig. 2c.

The four-dimensional analogue of the dodecahedron is a regular polytope called the 120-cell or the dodecaplex [10]. This polytope is made up of 120 regular dodecahedra and includes 720 pentagons (the interfaces between the dodecahedra), 1200 edges, and 600 vertices.

The Color Plate is a picture of this polytope once it has been pushed out to lie on the sphere in four-dimensional space, and then stereographically projected onto three-dimensional space. Again, because the right number of dodecahedra meet around edges and vertices, and because stereographic projection preserves spheres and angles, this complicated figure satisfies both the combinatorial and the curvature requirements to be a soap bubble cluster. Our transparent rendering of it incorporates both the Fresnel effect and interference patterns. Not all the bubbles in the center can be seen in a picture this size—they are very small just like the small pentagons in Fig. 2c.

Just as Fig. 2b in space has more symmetry than Fig. 2c in the plane, our cluster of soap bubbles has more symmetry in the sphere in four-dimensional space before the stereographic projection. If this sphere were rotated, the stereographically projected image in our cluster would swirl around. Each bubble would move and take the place of one of its neighbors [11].

We can start to understand the structure of this cluster by looking at the layers visible in the picture. The infinite region outside the cluster is the image of a dodecahedron near the north pole of the original sphere in four-space. It touches the 12 largest, round bubbles. Moving inwards, we next find a layer of 20 pointed bubbles, each fitting between three of the largest ones. Further in, there is another layer of 12 bubbles, the smallest ones easily visible. Next comes a middle layer of 30 bubbles, which come from the equatorial plane of the original sphere. Inside this, we find the same layers repeated in opposite order: layers of 12, then 20, then 12, surrounding a single small dodecahedral bubble coming from the south pole of the original sphere.

In our rendering of this large cluster, as well as the one in Fig. 1, we have chosen a point of view which is not along any of the many axes of symmetry. One can still see the symmetry of the overall object, but there is more information in the picture, since different parts are seen from different viewpoints. Also, part is rendered in front of a bright window, to better show the transparency, and part in front of a darker wall, to show off the highlights.

The Extra Color Plate shows the result of applying the same process to the hypercube, another regular polytope which projects to a soap bubble. The structure here is easier to see: a cube near the north pole in the hypercube projects to the central bubble, while the infinite outer region comes from a cube around the south pole. The other six cubes in the hypercube become the six bubbles surrounding the center.

**Where does this lead?**

The rapid development of computer graphics is especially exciting to mathematicians. It provides new ways to visualize mathematical objects known for a long time, and also, along with new computational tools, opens the door to new mathematical discoveries. Realization of these potentials will require new insights about the way we visualize geometric structures. This is an area in which artists may be able to help mathematicians. Hand drawings are superior to computer images for illustrating many ideas in geometry—especially in topology. The challenge is to program computers to extract the essential visual clues about a shape and to present these to the eye. Much modern geometry is done in higher dimensions, where visualization presents even more problems.

Advances in computation and graphics are changing the way mathematics is being done. As we build new mathematics on the foundations of the knowledge of several millennia, we must develop the potential inherent in computer-generated images and must find artistic methods of expressing mathematical visions.

**Acknowledgement**

We would like to thank S. Levy, M. Sullivan and J. Taylor for their valuable comments on the first draft of the present article, and P. Hanrahan for first suggesting an attempt to realistically model soap film with RenderMan. We have made use of software written by C. Gunn, K. Brakke and others at the Geometry Supercomputer Project.

## References and Notes


1. The expository paper by F. J. Almgren, "Minimal surface forms", *The Mathematical Intelligencer*, vol. 4, no. 4, 164–172 (1982), discusses all of these. Substantial portions of the classic work by D. W. Thompson, *On Growth and Form* (abridged edition edited by J. T. Bonner, Cambridge University Press, 1961) are devoted to geometries associated with minimal surface forms. The book of C. V. Boys, *Soap Bubbles: Their Colors and the Forces Which Mold Them* (New York: Dover 1959) is a classic of scientific exposition which gives a wonderful introductory appreciation and understanding of surface tension phenomena in soap films. The article by M. Emmer, "Soap Bubbles in Art and Science: From the Past to the Future of Math Art", *Leonardo*, vol. 20, no. 4, 327–334 (1987), discusses the history of soap bubbles in art.

2. J. E. Taylor, "The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces", *Annals of Mathematics* **103**, 489–539 (1976). See also F. J. Almgren and J. E. Taylor, "The geometry of soap films and soap bubbles", *Scientific American*, July 1976, 82–93, for a more informal, expository discussion.

3. The special case of finding oriented surfaces of least area spanning boundary curves is in somewhat better shape because of the work of H. R. Parks, "Explicit determination of area minimizing hypersurfaces, II", *Memoirs of the American Mathematical Society*, **60** Number 342, March 1986 and that of John Sullivan, "A Crystalline Approximation Theorem for Hypersurfaces", *Ph.D. thesis*, Princeton University, 1990 (also available as Research Report GCG 22 from the Geometry Center, 1300 South Second Street, Minneapolis, MN 55454). For unoriented least area surfaces, like Möbius bands, and for clusters including trapped volumes, the computational least area problem is not solved.

4. The Surface Evolver computer program was written by K. Brakke, and is available free of charge. Those with access to the Internet can get it by anonymous `ftp` to `geom.umn.edu`. The file `pub/evolver.tar.Z` is a compressed `tar` file including source code and documentation for its use (in the `.doc` files).

5. One of the most dramatic scenes in the motion picture *Soap Bubbles* (part of Michele Emmer's series on Mathematics and Art) shows real soap films rotating under bright lights to a musical accompaniment. The video tape, "Computing soap films and crystals", shows rotating transparent soap films computed by the Minimal Surface Team of the Geometry Supercomputer Project. To obtain a copy of this VHS format video tape, send a check for twenty dollars (payable to the Geometry Center) to Angie Vail, Geometry Center, 1300 South Second Street, Minneapolis, MN 55454.

6. See, for example, the stereoscopic pair of X-ray photographs of the interior grain structure of an aluminum-tin alloy reproduced on page 166 of Almgren [1].

7. First done by K. Perlin, "An Image Synthesizer", *Computer Graphics*, vol. 19, no. 3, 287–296 (1985), and later used by P. Hanrahan in the video tape [5].

8. Boys [1] gives a nice explanation without mathematical equations.

9. Striking pictures of real soap bubbles, showing these effects, were taken by science photographer F. Goro for *Scientific American* [2].

10. See H. S. M. Coxeter, *Regular Polytopes* (New York: Dover, 1973), for more discussion of this and other polytopes in higher dimensions. The article by John Sullivan, "Generating and Rendering Four-Dimensional Polytopes", *The Mathematica Journal*, vol. 1, no. 3 (1991), explains how we used the symmetry group of this polytope to compute (in the computer language *Mathematica*) the geometry for the Color Plate.

11. Such motion would show the covering translations of another beautiful and related mathematical object, the Poincaré dodecahedral space or homology three-sphere. This manifold can be obtained by identifying opposite faces of one of the bubbles in the cluster.

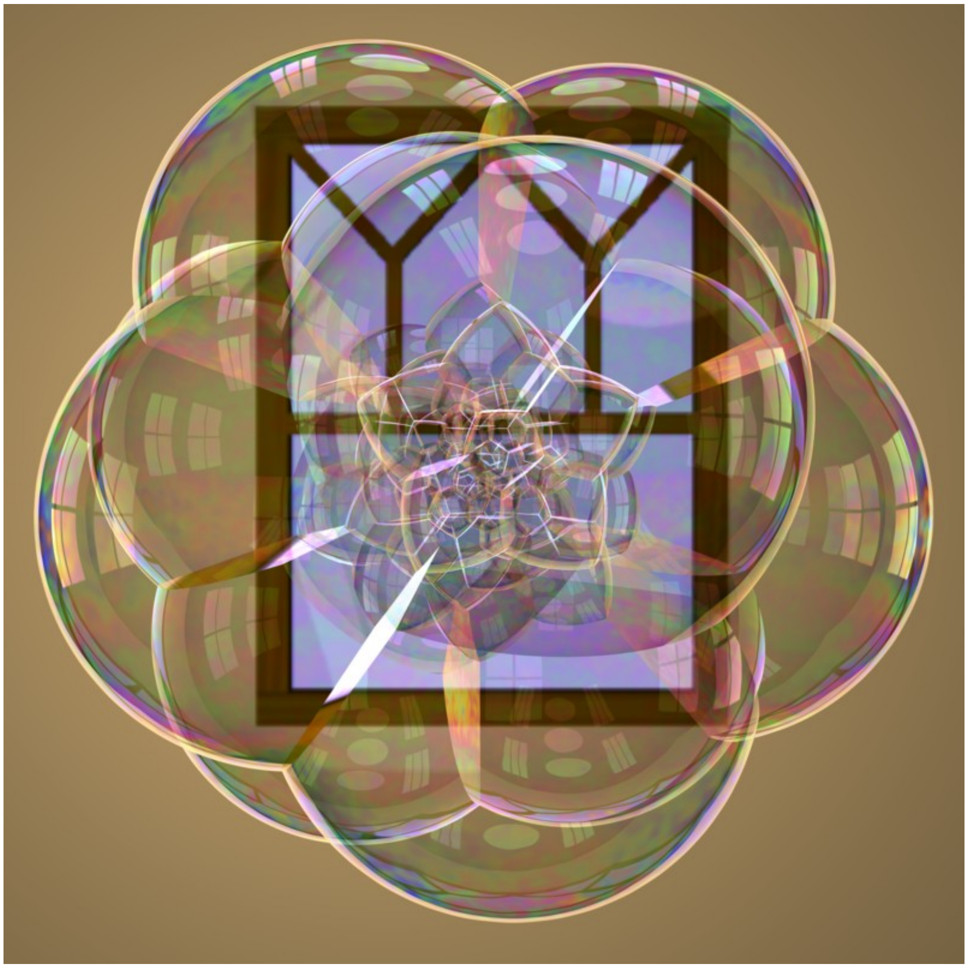
## Glossary

**Area-minimizing:** A surface is called area-minimizing if no other surface satisfying the same constraints has smaller area. Constraints in a typical problem might be that the surfaces span a boundary wire or enclose a certain volume.

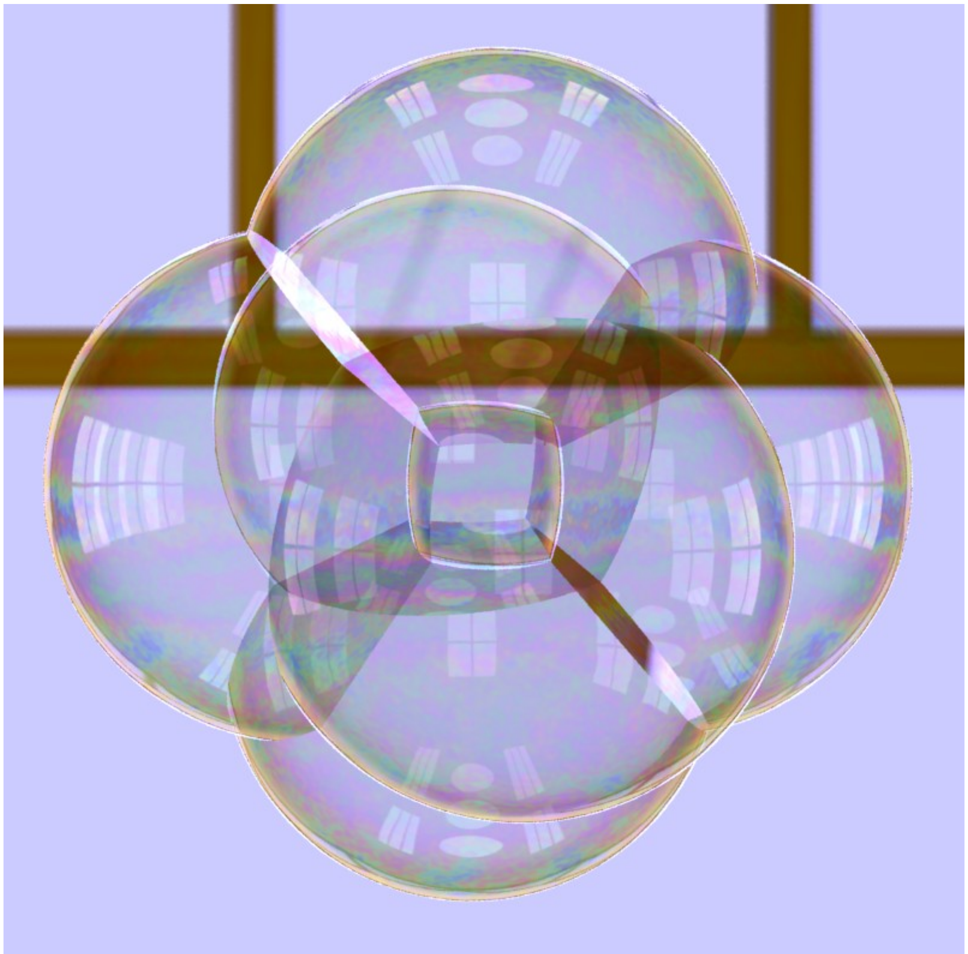
**Gaussian and mean curvature:** A surface has positive Gaussian curvature at points where it is convex or concave, and has negative Gaussian curvature where it curves different ways, like the seat of a saddle. The mean curvature at a point records the net direction of the surface's curvature and gives the force with which surface tension pulls. More precisely, at each point there is some direction in which the surface is most sharply curved. The curvatures in this direction and in the direction perpendicular to it are called principal curvatures; the average of these two curvatures is the mean curvature, and their product is the Gaussian curvature.

**Stereographic projection:** To make a map of a sphere (like the Earth) on a plane (like a piece of paper) we must introduce distortion somewhere. One possible map is made by stereographic projection, for which the south pole of the sphere is placed on the plane. The image of any point $p$ on the sphere is then the point where the line from the north pole through $p$ intersects the plane. Objects near the north pole are immensely stretched in the map compared to those near the south pole. But stereographic projection is conformal, which means that there is no distortion of angles at any point, and circles are taken to circles.

**Fresnel effect:** When light hits the boundary between two transparent materials (like air and glass or water), some is transmitted (or refracted) and the rest is reflected. The fractions depend on the angle of the incident light. One readily observable effect, called the Fresnel effect, is that more light is reflected at a shallow angle. This effect can be computed explicitly from the equations of electromagnetic waves known as Fresnel's Laws. These laws exactly describe the behavior of light at such an interface and thus also predict further effects which seem qualitatively different, such as the rainbow colors seen in thin films.



Color Plate. A large soap bubble cluster illustrating the computer graphics techniques which accurately model the Fresnel effect of decreased transparency at oblique angles and the colored interference patterns in a thin film. This is the most intricate soap bubble geometry which has been rendered by such techniques. It is the stereographic projection into three-dimensional space of a regular four-dimensional polytope called the dodecaplex. The soap films in this cluster are all pieces of spheres, and meet at the correct angles to be a physical soap film.



Extra Color Plate. A hypercube, when stereographically projected to three-dimensional space, forms this cluster of seven bubbles. The center bubble is a cube with spherical faces, meeting at 120° angles as sheets of soap film must. It was rendered by the techniques we describe, modeling the optics of thin films.

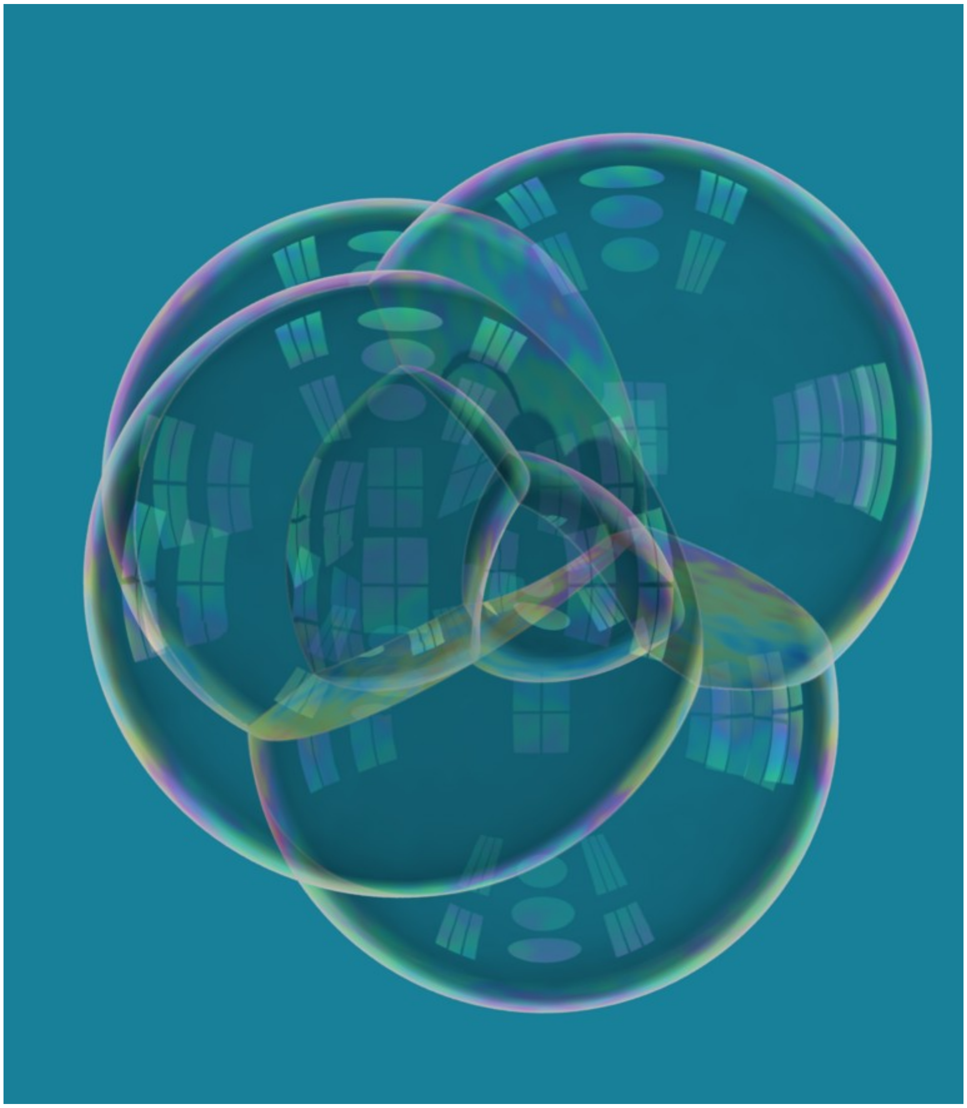
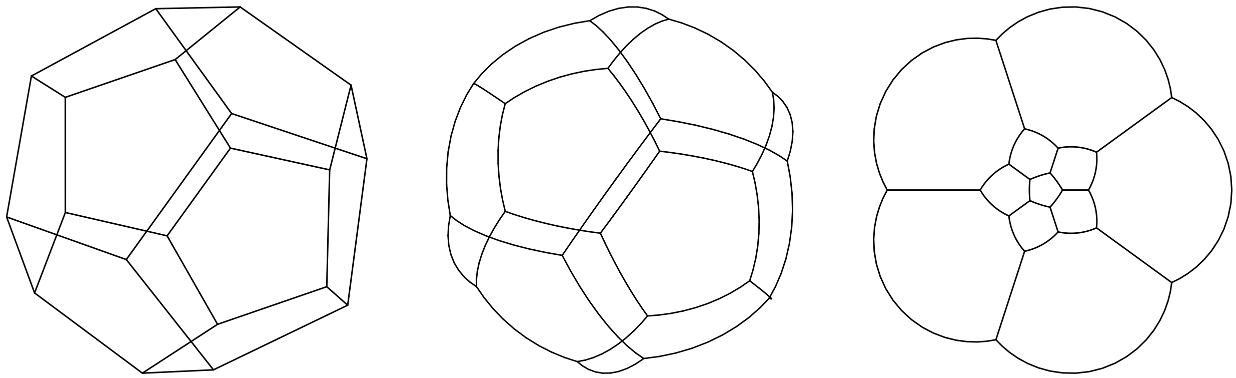


Figure 1. This cluster of six bubbles was generated by the Surface Evolver program. It was rendered by the computer graphics techniques described, which model the Fresnel effect of decreased transparency at oblique angles. Note the saddle-shaped interface in the middle. Among all soap bubble clusters presently known which have such a non-spherical interface, this one—discovered by John Sullivan—has the smallest number of regions of trapped air.



(a) (b) (c)

Figure 2. Part (a) shows the regular dodecahedron in space. It is the starting point for a the construction of two-dimensional analog to the three-dimensional soap bubble cluster illustrated in the Color Plate. Part (b) shows the central projection of the dodecahedron (a) onto a sphere. The arcs of this figure on the sphere are segments of great circles. Part (c) shows the spherical dodecahedron (b) projected stereographically onto a plane. The arcs are still segments of circles, and meet at the equal 120° angles of a bubble cluster.