POLYGON IN TRIANGLE: GENERALIZING A THEOREM OF POST

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Abstract. Steinhaus posed the problem of determining when one triangle P fits inside another one T; Post solved this by proving that if P fits somehow, it can always be placed with two vertices lying one the same edge of T. Here, we generalize and strengthen this result to show that any polygon P lying in a triangle T can be moved rigidly and continuously (staying inside T) until two of its vertices lie on the same edge of T. It follows that there are at most $6n$ ways to fit an n-gon into T.

A problem of Steinhaus [Ste] (repeated in [CFG]) asked for a characterization of which triangles P would fit into a given triangle T. Post [Pos] gave an answer in terms of 18 sets of inequalities in the side lengths, at least one of which must be satisfied for the triangle to fit. His proof relied on a theorem saying that if P fits in T somehow, then it will also fit in a position with two vertices lying on the same edge of T; his 18 sets of inequalities correspond to the 18 ways this can happen.

In this note, we generalize this theorem to an arbitrary polygon P, and strengthen it to show that P can be moved rigidly inside T from any starting position to the desired position. As a corollary, we deduce that there are at most $6n$ (disconnected) ways to fit a given n-gon into T. This bound is sharp for $n = 3$ and presumably also for larger n.

We will start by giving precise definitions, showing the equivalence of a few statements of the main theorem, and deriving corollaries. Then we will prove an important lemma, which has a different character from the rest of the argument. Finally we will give a proof of the main result.

Notation: Throughout, we will fix a triangle $T = \Delta OAB$, and a planar polygon P with n labeled vertices. The space of all polygons congruent to the fixed polygon P (which we will call *images* of P in the plane) is equivalent to the space of all Euclidean motions, and thus has a natural topological structure. (We have labeled the vertices of P in order to still distinguish all different images of P even when P is symmetric.) We will be interested in the configuration space $C(P, T)$, the subset consisting of those images of P contained in T. Its connected components are the different "ways" to place P inside T; two images of P in T are considered equivalent if one can be moved to the other staying inside T. Our main result will be the following:

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Theorem: *Each connected component of $C(P,T)$ contains an image of P where two vertices of P lie on the same edge of T. In other words, if a polygon P is contained in a triangle T, it can be rigidly moved, staying inside T, until two of its vertices lie on the same edge of T.*

Remark: Because a triangle T is convex, a polygon P is contained in T if and only if its vertices are all contained in T, thus if and only if its convex hull $\bar{P}$ is contained in T. So, by replacing P by $\bar{P}$ in the theorem we may assume that P is convex. In this case the conclusion can be restated more like Post's original theorem, and we see that our main theorem is equivalent to the following:

Theorem: *If P is a convex polygon, then each connected component of $C(P,T)$ contains an image of P where some edge of P is contained in an edge of T. In other words, if a convex polygon P is contained in a triangle T, it can be rigidly moved, staying inside T, until one of its edges lies along an edge of T.*

We can also consider all polygons similar to P which fit inside T; this idea was used in Post's original proof [Pos].

Proposition: *Among the polygons inside a triangle T which are similar to a given polygon P, any largest one sits rigidly inside T.*

Proof. Suppose Q is such a largest polygon similar to P fitting in T. First note that there must be a vertex of Q on each edge of T: if Q does not touch $\overline{AB}$ for instance, then a homothety around O can increase the size of Q until it does, contradicting maximality. (Note that if a vertex of Q is at a vertex of T it counts as being on both incident edges.)

Now, if Q does not fit rigidly (technically, we mean that a connected component of $C(Q,T)$ consists of more than a single point) consider a continuous path of images of Q in T. At each time, we can apply the argument in the previous paragraph, so there would still be a vertex of Q on each edge of T. But there is no rigid motion of the plane which can preserve this property: rigid motions which keep two vertices on two given edges $\overline{OA}$ and $\overline{OB}$ move other points along ellipses centered at O, as discussed in the lemma below. These can degenerate into segments, but only segments through O. □

Combining this with our main theorem gives an immediate corollary.

Corollary: *Among the polygons inside a triangle T which are similar to a given polygon P any largest one has two of its vertices along the same edge of T (and some vertex on each edge of T).*

Proof. Apply the main theorem to this largest similar polygon; it can be moved to a polygon Q with two vertices on one edge (say $\overline{AB}$) of T. But by the proposition just proved, Q sits rigidly in T, so that there can have been no motion involved, and the original largest copy also had the desired property. □

Remark: This corollary is equivalent to Post's version of the main theorem, that any polygon P in T has a congruent image with two vertices on one edge of T. To state a strengthened version, equivalent to our theorem, we would need to talk about a "locally largest" similar polygon, or a a largest similar polygon lying in T in the same way.

Finally, another easy corollary gives a bound on the number of ways a polygon can fit into a triangle.

Corollary: *An n-gon P can fit into a given triangle T in at most $6n$ ways ($3n$ in each orientation). That is, the configuration space $C(P, T)$ of images of P inside T has at most $6n$ components.*

Proof. Replacing P by its convex hull does not change the configuration space, and can only decrease the number of sides, so we may assume P is convex. Each component of $C(P, T)$ contains an image of P with one edge along an edge of T. Clearly all the images with a particular edge e along a given edge of T in the same orientation are connected: we can move one to another merely by translating along the edge in question. But there are n choices of edges of P, 3 choices of edges of T and 2 choices of orientation, giving at most $6n$ components. □

Remark: When P is a triangle, this bound is sharp, as shown by an example suggested by John Wetzel. Let T be an equilateral triangle, and P an isosceles triangle of the same height and apex angle $\pi/9$. The P fits in T either with its base along the base, or with one leg along an edge of T; in either configuration the apex of P is at a vertex of T and P fits rigidly. Of course the symmetries of these triangles mean that many of the 18 ways P fits in T look the same; we could avoid this by taking a slightly larger T and smaller P without symmetry.

Now we turn to the proof of the main theorem. We first give an important lemma which discusses the rigid motions which keep two given points on two given lines. We can imagine a physical device consisting of a writing surface with two intersecting straight tracks. A rigidly movable platform has two pegs (which sit in the tracks) and various pens at different positions relative to the pegs. As we move the pegs in their respective tracks, the lemma says that each pen traces out an ellipse centered at the intersection of the tracks. When we consider such a one-parameter family of motions, we will often call them "rotations", even though they are not rotations about a fixed center.

Lemma: *Let u, v and w be any three vertices of a polygon P, and consider all images of P for which u lies on a line $\overleftrightarrow{OA}$ and v lies on a different line $\overleftrightarrow{OB}$. The the locus of images of w is an ellipse (possibly degenerate) centered at O, and traced out monotonically.*

Remark: The polygon P does not enter here; u, v and w can be any three points in a plane which moves rigidly relative to the plane of ΔOAB. We will apply the lemma sometimes in the case $u = v$, in which case this point is fixed at O, and the locus of any other point under rotation about O is clearly a circle.

Proof. Let d be the distance between u and v. We will use as a reference an image where $v = O$, u is on $\overrightarrow{OA}$, and $w = w_0$; any other image is obtained by rotating clockwise around O by some angle θ and then translating along $\overleftrightarrow{OB}$ until u is on $\overleftrightarrow{OA}$ again. If ϕ is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then, as shown in Figure **??**, the distance we must translate is $d \sin\theta / \sin\phi$. Identifying the plane with the complex numbers so that $\overrightarrow{OB}$ is the positive real axis, we thus have $w(\theta) = e^{-i\theta} w_0 + \frac{d}{\sin\phi} \sin\theta$ which, having only linear terms in $\sin\theta$ and $\cos\theta$, is clearly the equation of an ellipse. □

Remark: The important point here is that an ellipse is convex; if there are some images of P for which w is on the opposite side of $\overleftrightarrow{AB}$ from O, then these form a single connected set. Thus if w hits $\overline{AB}$ when we rotate one way in the one-parameter family, it will not do so when we rotate the other way. It is tempting to look for a way to prove this needed part of the lemma without resorting to any calculations.

This lemma makes it quite straightforward to prove the main theorem.

Proof. Given an image of P in T, we can first translate it (in the direction $\overrightarrow{BO}$ for instance) until some vertex (call it u) lies on $\overline{OA}$. Next we can translate in the direction $\overrightarrow{AO}$ until some vertex (call it v) lies on $\overline{OB}$. (Note that we may have $u = v$ at O.)

Now, as in the lemma, there is a one-parameter family of motions (parameterized by a rotation angle) which keep u on $\overline{OA}$ and v on $\overline{OB}$. Try rotating clockwise in this family (translating, of course, as necessary to keep u and v on the desired lines) as far as possible with the image of P staying inside T. We will be forced to stop only when one of the following events happens:

(1) u or v reaches O (so u and v are on a common side of T),

(2) some other vertex reaches $\overline{OA}$ or $\overline{OB}$,

(3) u or v reaches A or B, or

(4) some other vertex (call it w) reaches $\overline{AB}$.

In the first two cases, we are clearly done. Note that the third case cannot arise when we start with $u = v = O$. If it does arise, we have a vertex of P now at a vertex of T, so we can repeat, calling these $u = v = O$, and this case will not recur.

In the fourth case, we repeat the same process but rotating in the other direction. We reach one of the same four cases; the first three are handled as before. But we still must deal with the case where we again have reached the last case, where some vertex w' has reached $\overline{AB}$.

The lemma says that a point w cannot rotate out of T in two different directions, so we see that w and w' are distinct vertices of P. Think of $\overline{AB}$ as a horizontal edge at the bottom of T, and consider the heights of different vertices of P as we rotate. When we have rotated all the way clockwise, w is a lowest vertex of P; when we have rotated all the way counter-clockwise, w' is a lowest vertex of P. Thus at some intermediate time, there are two equally lowest vertices of P (necessarily w and w' if these are adjancent in P). In this configuration, we can translate P towards $\overline{AB}$ (in the direction $\overrightarrow{OA}$ for instance) until these vertices both lie on $\overline{AB}$. $\square$

References

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