TIGHT KNOT VALUES DEVIATE FROM LINEAR RELATIONS

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Applications of knots to the study of polymers have emphasized geometric measures on curves such as 'energy' [1, 2, 3, 4] and 'rope length' [5, 6, 7], which, when minimized over different configurations of a knot, give computable knot invariants related to physical quantities [8]. In DNA knots, electrophoretic mobility appears to be correlated with the average crossing number of rope-length-minimizing configurations [9], and a roughly linear empirical relation has been observed between the crossing number and rope length [10]. Here we show that a linear relation cannot hold in general, and we construct infinite families of knots whose rope length grows as the $3/4$ power of the crossing number [11]. It can be shown that no smaller power is possible [12, 13, 14].

One measure of geometric complexity for a space curve γ is $A(\gamma)$, the average number of crossings in planar projections of γ. Another scale-invariant measure is rope length $L(\gamma)$, the quotient of arclength by thickness (the diameter of the largest uniform tube centered on γ).

If we fix a knot type K (or a link type K: our methods work for curves with one or many components) the infima (greatest lower bounds) of A and L over γ in K are the crossing number $C(K)$ and rope length $L(K)$ of K. A curve γ achieving the infimum $L(K)$ is called 'tight'. Experiments [9] with knotted DNA and simulations [10] of thermal averages of such knots lead us to seek mathematical relations between the invariants C and L, and between A and L for tight curves.

For any fixed power $3/4 \leq p \leq 1$ we have constructed infinite families of knots and of links that have $L \sim C^p$ — the ratio L/C^p is bounded above and below across each family. No $p < 3/4$ can work, because $C \leq A$ and there is a universal lower bound [12] for $L/A^{3/4}$. Furthermore, rope-length minimizers in a family with $L \sim C^{3/4}$ must satisfy $C \sim A$; thus these tight curves have $L \sim A^{3/4}$.

Any knot or link arises from a 'braid', a collection of N ascending arcs in a cylinder Z, joining N points on the bottom of Z to the same N points on the top. Z is bent into a solid torus S, the top and bottom disks are identified, and the arcs are joined to give a link in S. For a braid with bounds on its slope, curvature and horizontal strand separation, this bending fixes rope length within a uniform factor (depending on the shape of Z and S).

For a family of N-component Hopf links with $L \sim C^{3/4}$, we took cylinder Z of height h and radius r. N points are distributed on the bottom disk of Z, and rotated one full turn as they ascend. The resulting braid (Fig. 1a) has N helical strands of bounded slope and curvature provided $h \sim r$; to separate strands, we needed $r \sim N^{1/2}$. Thus the braid (and the corresponding link, Fig. 1b) has a rope length $L \sim hN \sim N^{3/2}$. Since each component links every other component exactly once, $C \geq N(N-1)$; the standard projection has $N(N-1)$ crossings, so $C = N(N-1) \sim N^2$.

For $(N, N-1)$-torus knots in S with $L \sim C^{3/4}$, we repeated this construction in the lower half of Z. To define the rest of the braid, we chose a circuit joining the N points (Fig. 1c) and slid each point along this circuit to the next while ascending through the upper half of Z. The horizontal distance a point travels is comparable to, at most, r. So as before, we needed $h \sim r \sim N^{1/2}$ and rope length $L \sim N^{3/2}$. The minimum crossing number of this knot [15] is $C = N(N-2) \sim N^2$. (Lattice knots with the same growth of crossing number have been found independently [16].) Many further families of links can be thus constructed [11].

If we plot C versus L on a log–log scale (Fig. 1d), then links in any family with $L \sim C^p$ approach the ray of slope p. We have seen examples with $p = 3/4$ (sublinear growth). To get $p = 1$ (linear growth), consider simply linked chains of N components, for which $L = 2\pi N + 2(N-2)$, whereas $C = 2(N-1)$ (ref. 11). Combining these examples yields families of links with any $3/4 \leq p \leq 1$.

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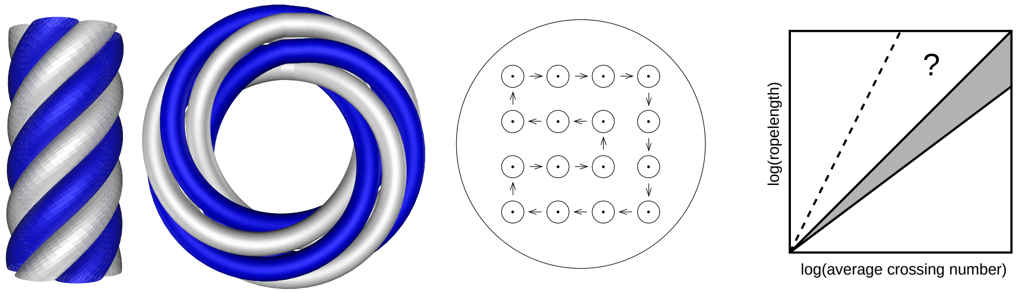

FIGURE 1. Hopf links with rope length L growing as the $3/4$ power of crossing number C. **a**, This 'braid' has seven helical strands, one hidden in the centre. **b**, The braid can be bent into this Hopf link. **c**, The construction can be modified by following this circuit to produce a torus knot instead. **d**, The general relation between crossings and rope length when viewed at large scale: for any family of links, the growth rate must lie between powers $3/4$ and 2, and all powers below 1 are realized.

It is unknown whether any family has superlinear growth. But if links with split, unknotted components are excluded, an argument based on embedding planar projections [11] proves no family has growth $p > 2$.

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