Math 302
Conformal Maps of $\mathbb{E}^2$ — Definitions and Theorems

Let $\gamma : (a, b) \to \mathbb{E}^2$ be a path. We will assume that $\gamma$ is differentiable. Expressed in coordinates, $\gamma(t) = (x, y)$.

Definition. Tangent vector: The tangent vector to the path $\gamma$ at a point $\gamma(t)$ is $\gamma'(t) = (x'(t), y'(t))$.

For example, if $\gamma(t) = (t, t^2)$, then the tangent vector at $t$ is $(1, 2t)$. At $t = 0$, for example, the tangent vector is $(1, 0)$.

Definition. Angle between two paths: The angle between two intersecting paths is the angle between their tangent vectors at the point of intersection.

For example, $F(x, y) = (3y, 3x)$ is conformal. It is not an isometry. $F(x, y) = (x^3, y)$ is not conformal.

Definition. A map is conformal at a point $P$ in its domain if it preserves the measures of all angles at the point $P$. (note: this does not mean that the map must send $P$ to itself.)

It is possible for a map to be conformal at some points but not at all points.

Definition. Conformal map: Let $F$ be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. $F$ is conformal if it is conformal at all points.

Definition. Distortion: (see page 169 of text) Let $\gamma$ be a path, and let $F : \mathbb{E}^2 \to \mathbb{E}^2$ be a map, not necessarily conformal. Then the distortion of $F$ along $\gamma$ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma) \text{ from } F(\gamma(t_0)) \text{ to } F(\gamma(t))}{\text{the length of the path } \gamma \text{ from } \gamma(t_0) \text{ to } \gamma(t)}.$$

In coordinates, this is

$$\frac{\| \frac{\partial F}{\partial x} \cdot x'(t_0) + \frac{\partial F}{\partial y} \cdot y'(t_0) \|}{\sqrt{(x'(t_0))^2 + (y'(t_0))^2}}.$$

This expression may also be written as a ratio of the lengths of tangent vectors:

$$\frac{\| (F\gamma)'(t_0) \|}{\| \gamma'(t_0) \|}.$$

Theorem. If a map is conformal at a point $P$, then at $P$, the distortion is the same in all directions.

Theorem. Conformal Criterion Theorem. Given a map $F$ and a point $P$ in the domain of $F$, suppose that there are two paths which intersect in a right angle at $P$. Suppose that the images under $F$ of the two paths intersect in a right angle. Suppose also that the distortion in the direction of these two paths is the same. Then $F$ is conformal at $P$.

Theorem. Corollary of the Conformal Criterion Theorem. Let $F$ be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. If, at a point $P$, the vectors $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ have the same length and are perpendicular to each other, then $F$ is conformal at $P$. Note that the length of $\frac{\partial F}{\partial x}$ is the distortion of $F$ in the horizontal direction and the length of $\frac{\partial F}{\partial y}$ is the distortion of $F$ in the vertical direction.
Conformal Maps from One Surface to Another — Definitions and Theorems

You will notice that the definitions and theorems in this handout are very similar to those in the handout *Conformal Maps of $\mathbb{E}^2$ — Definitions and Theorems*. Here, though, the setting is more general; the maps go from one surface to another, not necessarily $\mathbb{E}^2$. We do assume, when necessary, that the spaces are embedded in 3-space.

Let $\gamma : (a, b) \to M$ be a path. We will assume that $\gamma$ is differentiable. Expressed in coordinates, $\gamma(t) = (x, y, z)$.

**Definition.** Tangent vector: The tangent vector to the path $\gamma$ at a point $\gamma(t)$ is $\gamma'(t) = (x'(t), y'(t), z'(t))$.

**Definition.** Angle between two paths: The angle between two intersecting paths is the angle between their tangent vectors at the point of intersection.

**Definition.** A map is conformal at a point $P$ in its domain if it preserves the measures of all angles at the point $P$. (note: this does not mean that the map must send $P$ to itself.)

**Definition.** Conformal map: Let $F$ be a map from $M$ to $M'$. $F$ is conformal if it is conformal at all points.

**Definition.** Distortion: The distortion of $F$ at a point $P$ in a given direction is the amount of stretching in that direction.

We will now see a more precise definition and some formulas for distortion in the setting of $F : \mathbb{E}^2 \to S^2$.

**Definition.** Distortion: (see page 169 of text) Let $\gamma$ be a path, and let $F : M \to M'$ be a map, not necessarily conformal. Then the distortion of $F$ along $\gamma$ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma) \text{ from } F(\gamma(t_0)) \text{ to } F(\gamma(t))}{\text{the length of the path } \gamma \text{ from } \gamma(t_0) \text{ to } \gamma(t)}.$$

Now let $F : \mathbb{E}^2 \to S^2$, with coordinates $(u, v)$ on $\mathbb{E}^2$. Then $\partial F/\partial u$ is the image, under $F$, of the unit tangent vector in the horizontal direction and $\partial F/\partial v$ is the image, under $F$, of the unit tangent vector in the vertical direction. Therefore, the distortion of $F$ in the horizontal direction is $\|\partial F/\partial u\|$ and the distortion of $F$ in the vertical direction is $\|\partial F/\partial v\|$.

**Theorem.** Conformal Criterion Theorem. Given a map $F : M \to M'$ and a point $P \in M$, consider two paths which intersect in a right angle at $P$. Suppose that the images under $F$ of the two paths intersect in a right angle. Suppose also that the distortion in the direction of these two paths is the same. Then $F$ is conformal at $P$.

**Theorem.** Corollary of the Conformal Criterion Theorem. Let $F$ be a map from $\mathbb{E}^2$ to $S^2$, with coordinates $(u, v)$ on $\mathbb{E}^2$. If, at a point $P$, the vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ have the same length and are perpendicular to each other, then $F$ is conformal at $P$. 