Homework #7, Spring 2001

Due Wednesday March 21, at the beginning of class.

1. For a function $f$ from $\mathbb{E}^2$ to a surface, the factor by which $f$ distorts area is given, at each point, by the length of the cross product of $f_x$ and $f_y$. If $f_x$ and $f_y$ are perpendicular, this factor turns out to be the product of the lengths of $f_x$ and $f_y$. For the questions below, let the radius of the sphere be 1.

(a) Show that the cylindrical projection is area preserving.

(b) Is the Mercator projection area preserving? If not, compute the factor by which it distorts area.

(c) On the Mercator projection, which countries look too big (relative to other countries) and which look too small? Explain this in terms of your answer to part (b). Note: a copy of a Mercator projection of the globe is included on the hard copy given out in class.

2. Compute the distortion of each of the following maps:

   (a) Inversion in the unit circle:
   \[ i(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right). \]

   (b) Dilation by a factor of $r$: $f_r(x, y) = (rx, ry)$. (Here $r$ can be any positive constant.)

3. Let $i$ and $f_r$ be the functions given in the previous problem. Let $i_r$ be inversion in a circle of radius $r$ centered at the origin. Prove that $i_r = f_r \circ i \circ f_{1/r}$.

   Hint: What is the definition of $i_r$? Given any point $P$, let $s$ be the distance from $P$ to the origin. How far is $f_{1/r}(P)$ from the origin? How about $i(f_{1/r}(P))$? How about $f_r(i(f_{1/r}(P)))$? On what ray does each of these points lie?