Math 302

Conformal Maps of $\mathbb{E}^2$: Definitions and Theorems

Let $\gamma : [a, b] \to \mathbb{E}^2$ be a path. We will assume that γ is differentiable. Expressed in coordinates, $\gamma(t) = \big(x(t), y(t)\big)$.

Definition (tangent vector)**.** The *tangent vector* to γ at a point $\gamma(t)$ is $\gamma'(t) = \big(x'(t), y'(t)\big)$.

For example, if $\gamma(t) = (t, t^2)$, then the tangent vector at $\gamma(t)$ is $(1, 2t)$. At $t = 0$, for example, the tangent vector is $(1, 0)$.

Definition (angle between two paths)**.** The *angle* between two intersecting paths is the angle between their tangent vectors at the point of intersection.

Definition. A map is *conformal at* a point P in its domain if it preserves the measures of all angles at the point P. (note: this does not mean that the map must send P to itself.)

It is possible for a map to be conformal at some points but not at all points.

Definition (conformal map)**.** Let F be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. F is *conformal* if it is conformal at all points.

For example, $F(x, y) = (3y, 3x)$ is conformal (everywhere), but is not an isometry. $G(x, y) = (x^3, y^3)$ is not conformal, though it is conformal at some points, like $(1, 1)$.

Definition (distortion: see page 169 of text)**.** Let γ be a path, and let $F : \mathbb{E}^2 \to \mathbb{E}^2$ be a map, not necessarily conformal. Then the distortion of F along γ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma) \text{ from } F(\gamma(t_0)) \text{ to } F(\gamma(t))}{\text{the length of the path } \gamma \text{ from } \gamma(t_0) \text{ to } \gamma(t)}.$$

In coordinates, this is

$$\frac{\left\| \frac{\partial F}{\partial x} \cdot x'(t_0) + \frac{\partial F}{\partial y} \cdot y'(t_0) \right\|}{\sqrt{x'(t_0)^2 + y'(t_0)^2}}.$$

This expression may also be written as a ratio of the lengths of tangent vectors:

$$\frac{\left\| (F\gamma)'(t_0) \right\|}{\left\| \gamma'(t_0) \right\|}.$$

Theorem. *If a map is conformal at a point P, then at P, its distortion is the same in all directions.*

Theorem (Conformal Criterion)**.** *Given a map F and a point P in the domain of F, consider two paths which intersect at right angles at P. Suppose that the images under F of these paths intersect at right angles, and also that the distortions in the directions of these two paths are the same. Then F is conformal at P.*

Theorem (Corollary of Criterion)**.** *Let F be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. If, at a point P, the vectors $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ have the same length and are perpendicular to each other, then F is conformal at P.*

Note that the length of $\frac{\partial F}{\partial x}$ is the distortion of F in the horizontal direction and the length of $\frac{\partial F}{\partial y}$ is the distortion of F in the vertical direction.

Math 302

Conformal Maps from One Surface to Another: Definitions and Theorems

You will notice that the definitions and theorems on this page are very similar to those on the previous page (Conformal Maps of $\mathbb{E}^2$). Here, though, the setting is more general; the maps go from one surface to another, not necessarily $\mathbb{E}^2$. We do assume, when necessary, that the surfaces are embedded in 3-space $\mathbb{R}^3$.

Let $\gamma : [a, b] \to M$ be a path. We will assume that γ is differentiable. Expressed in coordinates, $\gamma(t) = \big(x(t), y(t), z(t)\big)$, where these are the usual coordinates on $\mathbb{R}^3 \supset M$.

Definition (tangent vector)**.** The *tangent vector* to γ at $\gamma(t)$ is $\gamma'(t) = (x'(t), y'(t), z'(t))$.

Definition (angle between two paths)**.** The *angle* between two intersecting paths is the angle between their tangent vectors at the point of intersection.

Definition. A map is *conformal at* a point P in its domain if it preserves the measures of all angles between paths at the point P. (Note: this does not mean that the map must send P to itself; the map may be from one surface to another.)

Definition (conformal map)**.** Let F be a map from M to M'. F is *conformal* if it is conformal at all points.

Definition (Distortion — see page 169 of text)**.** The *distortion* of F at a point P in a given direction is the amount of stretching in that direction, that is, the factor by which the length of a path through P is stretched at that point.

Let γ be a path, and let $F : M \to M'$ be a map, not necessarily conformal. Then the *distortion* of F along γ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma) \text{ from } F(\gamma(t_0)) \text{ to } F(\gamma(t))}{\text{the length of the path } \gamma \text{ from } \gamma(t_0) \text{ to } \gamma(t)}.$$

We will now give formulas for distortion in the setting of a map $F : \mathbb{E}^2 \to M'$; we will use rectangular coordinates (u, v) on $\mathbb{E}^2$. Then $\partial F/\partial u$ is the image under F of the unit tangent vector in the horizontal direction and $\partial F/\partial v$ is the image under F of the unit tangent vector in the vertical direction. Therefore, the distortion of F in the horizontal direction is $\|\partial F/\partial u\|$ and the distortion of F in the vertical direction is $\|\partial F/\partial v\|$.

Theorem (Conformal Criterion)**.** *Given a map $F : M \to M'$ and a point $P \in M$, consider two paths which intersect at P at right angles. Suppose that the images under F of the two paths intersect at right angles, and also that the distortions in the directions of these two paths are the same. Then F is conformal at P.*

Theorem (Corollary of the Criterion)**.** *Let F be a map from $\mathbb{E}^2$ to $M' \subset \mathbb{R}^3$. Use coordinates (u, v) on $\mathbb{E}^2$, and (x, y, z) on $M' \subset \mathbb{R}^3$. If, at a point P, the vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ (computed as for any map $\mathbb{E}^2 \to \mathbb{R}^3$) have the same length and are perpendicular to each other, then F is conformal at P.*