

## Congruence of Rays, Segments and Angles

Given any space  $M$  satisfying the protractor and mirror axioms, we know that we can compose reflections to give many other isometries. In particular, this will be true on  $\mathbb{S}^2$ ,  $\mathbb{H}^2$  and  $\mathbb{E}^2$ , where the protractor and mirror axioms do hold.

We now check that these isometries are sufficient to produce congruences between rays, between segments of equal length, or between angles of equal measure. (Remember, two geometric figures are called congruent if there is an isometry taking one to the other.)

**Theorem** (RCT: Ray Congruence Theorem). *Any two rays are congruent.*

*Proof.* Given two rays  $\overrightarrow{AB}$  and  $\overrightarrow{A'B'}$  we want to find an isometry taking one to the other.

First, let  $f$  be an isometry taking  $A'$  to  $A$ . If  $A' = A$ , then  $f$  can be the identity. Otherwise we can either let  $f$  be reflection across the perpendicular bisector of  $\overline{AA'}$ , or let  $f$  be translation along  $\overrightarrow{AA'}$  by distance  $AA'$ . Write  $B'' := f(B')$ .

We now want to find an isometry  $g$  such that  $g(\overrightarrow{AB''}) = \overrightarrow{AB}$ . Clearly,  $g$  should fix  $A$ . If the rays are already equal, let  $g$  be the identity. Otherwise, we can either let  $g$  be reflection across the angle bisector of  $\angle BAB''$  or let  $g$  be rotation around  $A$  by angle  $m\angle BAB''$ .

The desired isometry is now  $g \circ f$ , since  $g(f(\overrightarrow{A'B'})) = \overrightarrow{AB}$ . □

**Theorem** (SCT: Segment Congruence Theorem). *Two segments are congruent if and only if they have equal length. In symbols,*

$$AB = A'B' \iff \overline{AB} \cong \overline{A'B'}$$

*Proof.* If two segments are congruent, they must have equal length, since isometries preserve length. To show the converse, we must construct the desired isometry. By RCT, there is an isometry taking  $A'$  to  $A$  and  $\overrightarrow{A'B'}$  to  $\overrightarrow{AB}$ ; it must map  $B'$  to  $B$  because  $AB = A'B'$ . □

**Theorem** (ACT: Angle Congruence Theorem). *Two angles have equal measure if and only if they are congruent. In symbols,*

$$m\angle BAC = m\angle B'A'C' \iff \angle BAC \cong \angle B'A'C'$$

*Proof.* If two angles are congruent then, since isometries preserve angle measure, their measures must be equal.

For the other direction, we must construct the desired isometry. By RCT, there is an isometry  $f$  taking  $\overrightarrow{A'B'}$  to  $\overrightarrow{AB}$ . It takes  $C'$  to some point  $C''$ . If  $C''$  and  $C$  are on the same side of  $\overrightarrow{AB}$ , we are done. (By the protractor axiom there is a unique ray in that half-space making angle  $m\angle BAC$  with  $\overrightarrow{AB}$ .) Otherwise, reduce to this case by reflecting across  $\overrightarrow{AB}$ .

That is, the isometry we originally desired is either  $f$  or  $f$  composed with this reflection. It takes  $\angle B'A'C'$  to  $\angle BAC$ . □

Given an isometry  $f$  taking  $\overrightarrow{A'B'}$  to  $\overrightarrow{AB}$ , note that these proofs in fact show slightly more. Namely,  $f$  takes  $\overline{A'B'}$  to  $\overline{AB}$  if and only if these have equal length. Similarly,  $f$  takes  $\overrightarrow{A'C'}$  to  $\overrightarrow{AC}$  if and only if it takes  $C'$  to the  $C$ -side of  $\overrightarrow{AB}$  and  $m\angle BAC = m\angle B'A'C'$ .

Note that, when we work on the sphere, a pair of points  $A, B$  does not determine a unique segment  $\overline{AB}$  or ray  $\overrightarrow{AB}$ . In reading the results above, think of  $\overline{AB}$  as meaning a particular choice of segment, and  $AB$  as meaning the length of that segment, etc.

Note that when we concluded  $\overline{AB} \cong \overline{A'B'}$  we actually got an isometry taking  $A'$  to  $A$  and  $B'$  to  $B$ , not vice versa. And  $\angle ABC \cong \angle A'B'C'$  meant an isometry taking  $\overrightarrow{B'A'}$  to  $\overrightarrow{BA}$  and  $\overrightarrow{B'C'}$  to  $\overrightarrow{BC}$ , not vice versa. This is a minor point, since of course there are in both cases isometries that flip things, so  $\overline{AB} \cong \overline{BA}$  and  $\angle ABC \cong \angle CBA$ , but this convention will be useful.